Unit 1 · Permutation and Combination

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 29

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 "At least one": a count that came out too big

ANSWER HA · 5

Tests: overcounting when "at least one" is met by choosing one first; a wrong count explained by the exact factor.

a. All committees less those with no girl: $C(10, 4) - C(6, 4) = 210 - 15 = \mathbf{195}$.

b. Her first step names one girl as "the chosen one". A committee with girls Asmita and Bina is reached once with Asmita chosen first and once with Bina: **twice**. A committee with exactly k girls is counted k **times**, once for each of its girls.

c. Committees with exactly k girls: $C(4, k)C(6, 4 - k)$, that is 80, 90, 24 and 1 for $k = 1, 2, 3, 4$ (they add up to 195). Counted k times each:
$1 \times 80 + 2 \times 90 + 3 \times 24 + 4 \times 1 = 80 + 180 + 72 + 4 = 336$. ✓

(The tempting reply to a: "336 is right, the condition is met." The condition is met, but the committees are not counted once each.)

MARKING SCHEME

✓ a. $C(10, 4) - C(6, 4)$.
✓ a. 195.
✓ b. A committee is counted once for each girl on it: k times (2 for exactly 2).
✓ c. The numbers with exactly 1, 2, 3, 4 girls: 80, 90, 24, 1.
✓ c. $80 + 180 + 72 + 4 = 336$.

2 A round table with two conditions

ANSWER A · 5

Tests: a circular arrangement with a second condition; fixing the right person first; checking a count by a second route.

a. Seat the 5 Nepalis round the table: $(5 - 1)! = 24$ ways. That leaves 5 seats, one between each two Nepalis, and the table is now fixed, so the Indians fill them in $5! = 120$ ways: $24 \times 120 = \mathbf{2880}$.

b. The Nepalis as before: 24 ways. Arjun must take one of the 2 Indian seats beside Pemba: 2 ways. The other 4 Indians fill the 4 seats left: $4! = 24$. So $24 \times 2 \times 24 = \mathbf{1152}$.

c. However the Nepalis sit, Arjun can be in any of the 5 Indian seats, and each choice leaves the same 4! ways for the others, so the seatings are shared equally among his 5 seats. Exactly 2 of those seats are next to Pemba, so $\frac{2}{5}$ of the 2880 seatings: $\frac{2}{5} \times 2880 = 1152$. ✓

d. In a row the chairs are all different. Either a Nepali or an Indian takes the first chair (2 patterns), then 5! for each country: $2 \times 120 \times 120 = \mathbf{28{,}800}$. (Ten times the round table's count, since a seating round the table can be cut open at any of 10 places.)

MARKING SCHEME

✓ a. Nepalis 4!, then Indians 5! in the fixed gaps.
✓ a. 2880.
✓ b. Arjun in one of the 2 seats beside Pemba, the rest 4!: 1152.
✓ c. Arjun's 5 possible seats are equally shared and 2 of them are beside Pemba.
✓ d. $2 \times 5! \times 5! = 28{,}800$.

3 Routes through the old town

ANSWER

Tests: routes as words of E's and N's (1.3), "and then" multiplies, and the difference between closing a point and closing one street.

a. A route is a word of 6 E's and 4 N's: $\frac{10!}{6!\,4!} = \mathbf{210}$.

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b. A to P (2 E, 2 N): $\frac{4!}{2!\,2!} = 6$; and then P to B (4 E, 2 N): $\frac{6!}{4!\,2!} = 15$. So $6 \times 15 = \mathbf{90}$. A · 5

c. $210 - 90 = \mathbf{120}$.

d. No. Only the routes that walk from P straight on to Q are lost, not every route through P. Those go A to P (6 ways), the lane PQ, then Q to B (3 E, 2 N: $\frac{5!}{3!\,2!} = 10$): $6 \times 10 = 60$. So **150** routes are left, more than in c: the routes through P that turn north there are still open.

MARKING SCHEME

- ✓ a. 210.
- ✓ b. $6 \times 15 = 90$.
- ✓ c. 120.
- ✓ d. Kiran is wrong: only routes using the lane PQ are lost.
- ✓ d. $6 \times 10 = 60$ lost, 150 left.

4 Groups with and without names

ANSWER HA · 6

Tests: the "swap them, is it a different result?" test applied to whole groups; labelled and unlabelled groups; a count checked by a second route.

a. Choose 4 for water, then 4 of the 8 left for air, and the last 4 do soil: $C(12, 4)\,C(8, 4)\,C(4, 4) = 495 \times 70 \times 1 = \mathbf{34{,}650}$.

b. Without topics, the same three groups given out in a different order of topics is the same split. Each split comes from $3! = 6$ of the ways in a, so $34{,}650 \div 6 = \mathbf{5775}$. (The tempting answer: 34,650 again, as if the groups had names.)

c. Put Tashi and Gaurav together and choose the other 2 in their group from the 10 left: $C(10, 2) = 45$. The other 8 make two groups of 4 with no names: $C(8, 4) \div 2 = 35$. So $45 \times 35 = \mathbf{1575}$.

d. With Tashi in a group, there are 3 places left beside him out of the 11 places the other students fill, and every place is equally likely to be Gaurav's. So $\frac{3}{11}$ of the splits put them together: $\frac{3}{11} \times 5775 = 1575$. ✓

MARKING SCHEME

- ✓ a. $C(12, 4)\,C(8, 4) = 34{,}650$.
- ✓ b. The groups can be given out in 3! orders that make the same split.
- ✓ b. 5775.
- ✓ c. $C(10, 2) = 45$ for their group.
- ✓ c. $45 \times 35 = 1575$.
- ✓ d. Gaurav takes one of the 11 places, 3 beside Tashi: $\frac{3}{11} \times 5775 = 1575$.

5 Adding up all the numbers

ANSWER HA · 4

Tests: counting how often each digit sits in each place, instead of counting numbers; removing the strings that start with 0.

a. There are $4! = 24$ numbers. Fix a digit in one place: the other 3 digits fill the rest in $3! = 6$ ways, so each digit is in each place 6 times. Each place adds $6 \times (1 + 2 + 3 + 4) = 60$, so the sum is $60 \times (1000 + 100 + 10 + 1) = 60 \times 1111 = \mathbf{66{,}660}$.

b. First count all 24 strings of 0, 1, 2, 3, including those that start with 0: each place adds $6 \times (0 + 1 + 2 + 3) = 36$, so $36 \times 1111 = 39{,}996$. The strings that start with 0 are really the three-digit numbers made from 1, 2, 3: each digit is in each of their 3 places $2! = 2$ times, so their sum is $2 \times 6 \times 111 = 1332$. The four-digit numbers add up to $39{,}996 - 1332 = \mathbf{38{,}664}$.

MARKING SCHEME

- ✓ a. Each digit is in each place 3! = 6 times.
- ✓ a. $60 \times 1111 = 66{,}660$.
- ✓ b. All 24 strings: 39,996; those starting with 0 add 1332.
- ✓ b. 38,664.

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6 Fruits that are alike

ANSWER A · 4

Tests: when "in or out for each object" fails (alike objects); the basic principle of counting used on how many of each kind.

a. 2^9 treats the fruits as different: taking the first apple and taking the second give the same selection, "one apple", but are counted as two. With alike fruits, a selection is only how many of each kind.

b. Apples: 0, 1, 2 or 3 (4 ways); oranges 0 to 4 (5 ways); mangoes 0 to 2 (3 ways). "And then" multiplies: $4 \times 5 \times 3 = 60$, less the empty selection: **59**.

c. Now at least 1 of each: $3 \times 4 \times 2 = \mathbf{24}$.

MARKING SCHEME

✓ a. 2^9 counts the alike fruits as different (two single apples counted twice).

✓ b. $(3 + 1)(4 + 1)(2 + 1) = 60$.

✓ b. Less the empty selection: 59.

✓ c. $3 \times 4 \times 2 = 24$.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	14	15	**29**

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