Unit 15 · System of Linear Equations

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Computational Methods (Units 15–16) **or** Mechanics (Units 17–18), as the school teaches: 6 of 75 marks; 1 MCQ and 1 short question; no long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's model paper:** the short question split: 2 marks, "solve by Gauss–Seidel method" (two equations); then 3 marks, a simplex problem (Unit 16)
- **MCQ (1 mark)**
  - what Gauss elimination aims for
  - which values make y
  - the operation that removes x
  - which change alters the solutions
  - the dominant order
  - why partial pivoting
  - why it converges
  - one iteration's values
  - back substitution
  - k for infinitely many
  - the wrong order: values run away
- **Short (5 marks, in parts)**
  - Gauss elimination, three unknowns
  - Gauss–Seidel, two unknowns: arrange first, say why
  - Gauss–Seidel, three unknowns, to two decimal places
  - a heated plate: both methods, compared

Key terms

Augmented matrix: the coefficients, a bar, the right-hand sides; one row per equation.

Upper triangular: zeros below the diagonal; each equation one unknown fewer.

Back substitution: solving from the last equation up.

Pivot: the entry used to clear its column below it; **multiplier** m = entry ÷ pivot.

Partial pivoting: swap the row with the largest entry (in size) into the pivot place.

Iteration: one round improving every unknown once.

Diagonally dominant: in each equation the diagonal coefficient is larger in size than the others together.

Gauss elimination

Row operations (never change the solutions):

1. swap two rows
2. multiply a row by $k \neq 0$
3. $R_i \to R_i - mR_j$

Steps:

1. pivot $a_{11} \neq 0$; $m = a_{i1}/a_{11}$; $R_i \to R_i - mR_1$: x gone below row 1
2. pivot a_{22}: y gone from row 3
3. upper triangular: back-substitute, last unknown first
4. check in an original equation

- $x + y + z = 6$, $2x - y + 3z = 9$, $3x + 2y - z = 4$
  - $R_2 - 2R_1$, $R_3 - 3R_1$; then $R_3 - (1/3)R_2$
  - last row $(0\ 0\ -13/3 \mid -13) \to z = 3$, $y = 2$, $x = 1$
- **zero pivot**: swap with a row below that has a nonzero entry there
- **tiny pivot**: a huge multiplier magnifies rounding; computers pivot on the largest
- **last row** $(0\ 0\ 0 \mid c)$
  - $c \neq 0$: $0 = c$, **no solution**
  - $c = 0$: two equations, three unknowns, **infinitely many**
- Kirchhoff's laws → three equations in the currents; $I_1 = 2$, $I_3 = 3$, $I_4 = 1$ A

- the determinant = the product of the pivots

Gauss–Seidel

Each equation gives its own unknown:

$$x = \frac{d_1 - b_1 y - c_1 z}{a_1}$$
$$y = \frac{d_2 - a_2 x - c_2 z}{b_2}$$
$$z = \frac{d_3 - a_3 x - b_3 y}{c_3}$$

1. arrange for diagonal dominance (reorder the equations)
2. start from (0, 0, 0) unless told
3. x from the first; the **new** x at once into y; both into z
4. repeat until two iterations agree to the places asked

- $5x + y = 7$, $x - 4y = -7$: $x = (7 - y)/5$, $y = (x + 7)/4$
  - iteration 1: $x = 1.4$, $y = 2.1$; iteration 2: 0.98, $1.995 \to (1, 2)$
- **dominant**: $|a_1| > |b_1| + |c_1|$, $|b_2| > |a_2| + |c_2|$, $|c_3| > |a_3| + |b_3|$
  - sufficient, not necessary
  - $x - 4y = -7$ has the big y: it must give y
  - in the wrong order the values run away
- $6x + y - z = 10$, $x + 5y + 2z = 13$, $2x - y + 7z = 24$: dominant ($6 > 2$, $5 > 3$, $7 > 3$)
  - iteration 1: 1.667, 2.267, 3.276; by iteration 6: (2, 1, 3)
- **Jacobi**: uses only the last iteration's values; slower
- say where you start and when you stop

Which method

	Gauss elimination	Gauss–Seidel
answer	exact (up to rounding)	approximate, to the places asked
steps	fixed, about $(2/3)n^3$ operations	until it agrees
needs	nonzero pivots (swap)	diagonal dominance
good for	any system; says which case	big systems with few nonzero coefficients
a slip	carried to the end	corrected by later iterations

- Cramer by cofactors: $(n + 1) \times n!$ terms; 30 unknowns $\approx$ 120 million years on the fastest computer; elimination: 18,000 operations
- TOP500 ranks the fastest computers by how fast they solve a linear system
- heated plate: each point = the average of its four neighbors $\to 4T_1 - T_2 = 120$…

COMMON MISTAKES

✗ Gauss–Seidel with the old x when finding y
✓ the newest x, at once.

✗ Gauss–Seidel in the order given, not dominant
✓ reorder first; say why.

✗ multiplier = pivot ÷ entry
✓ entry ÷ pivot.

✗ a zero pivot divided by
✓ swap rows.

✗ (0 0 0 | 0) read as no solution
✓ infinitely many; (0 0 0 | $c \neq 0$) is none.

✗ stopping after a fixed number of iterations
✓ when two agree to the places asked.

✗ back substitution from the top
✓ from the last row up.