Unit 14 · Differential Equations

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Calculus (Units 12–14 together): 22 of 75 marks; 4 MCQs, 2 short and 1 long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's model paper:** short and long parts: write an equation of the first order and first degree (1 mark); solve a separable equation (2 marks); is it homogeneous? solve it (3 marks, higher ability)
- **MCQ (1 mark)**
  - order and degree (two)
  - the integrating factor
  - $dy/dx = y/x$
  - $y = vx$ in an equation
  - which one is exact
  - $dy/dx = e^{x+y}$
  - a half-life from λ
  - a curve from its slope
  - slope twice that of OP
  - Newton's cooling: the next time
- **Short and long (5 or 8 marks, in parts)**
  - order and degree; check a general solution
  - separable, with a condition
  - homogeneous: $y + xe^{y/x}$; is it homogeneous? why
  - linear: $\cos x\, y' + y \sin x = 1$
  - show exact; solve
  - cesium-137; boiled milk
  - a curve from a condition on its slope
  - linear in x and exact, both ways
  - a capacitor discharging and charging
  - Nepal's censuses: k, a prediction, a year

Key terms

Differential equation: an equation containing the derivatives of an unknown function.

Order: the highest derivative in it.

Degree: the power of that derivative, once free of roots and fractional powers of derivatives; not defined if a derivative is inside sin, log…

General solution: with as many arbitrary constants as the order; a family of curves.

Particular solution: one picked by a condition ($y = 2$ at $x = 0$).

Direction field: a dash of slope $f(x, y)$ at each point; solutions follow the dashes.

Homogeneous function of degree n: $f(tx, ty) = t^n f(x, y)$.

Integrating factor (IF): the multiplier that makes a linear equation's left side one derivative.

Exact: $M\,dx + N\,dy$ is du for some $u(x, y)$.

Partial derivative $\partial u/\partial x$: differentiate by x, y held constant.

Order, degree, forming

- $y'' + 9y = 0$: order 2, degree 1
- $y = xy' + \sqrt{1 + (y')^2}$: clear the root → $(y - xy')^2 = 1 + (y')^2$: order 1, **degree 2**
- $y'' = \sin(y')$: order 2, degree not defined
- **forming**: n constants → differentiate n times, eliminate them → order n
  - $y = Cx + C^2$: $C = y' \to y = xy' + (y')^2$
  - $y = A\cos 2x + B\sin 2x \to y'' + 4y = 0$
- **checking** a solution: substitute; it must hold for every C
  - $y = x - 1 + Ce^{-x}$ in $y' + y = x$; through $(0, 2)$: $C = 3$

Four types (first order, first degree)

$dy/dx = f(x, y)$, or $M\,dx + N\,dy = 0$

Type	Looks like	First move
separable	$dy/dx = f(x)g(y)$	y's with dy, x's with dx
homogeneous	$dy/dx = F(y/x)$	$y = vx$
linear	$dy/dx + Py = Q$	multiply by $e^{\int P\,dx}$
exact	$M\,dx + N\,dy = du$	test $\partial M/\partial y = \partial N/\partial x$

- of more than one type ($dy/dx = y/x$): use the easiest

Separable

- $g(y)\,dy = f(x)\,dx \to \int g(y)\,dy = \int f(x)\,dx + C$, **one** constant
- why allowed: $(G(y))' = g(y)y' = F'(x)$, so $G(y) - F(x)$ is constant
- write $\ln|N|$; $\pm e^c$ becomes one constant A
- dividing by y may **lose** the solution $y = 0$: check

- $dy/dx = f(ax+by+c)$: put $v = ax+by+c$
  - $dy/dx = (x+y)^2$: $v' = 1+v^2 \to \tan^{-1}(x+y) = x+C$

$N = N_0 e^{-\lambda t}$	$dN/dt = -\lambda N$ **Use:** decay
$T_{1/2} = \ln 2/\lambda$	the same for any amount **Use:** half-life
$P = P_0 e^{kt}$	$dP/dt = kP$; doubles every $\ln 2/k$ **Use:** growth
$T = T_s + (T_0 - T_s)e^{-kt}$	$dT/dt = -k(T - T_s)$ **Use:** Newton's cooling

- iodine-131, $T_{1/2} = 8.02$ days: after 30 days $\approx 7.5\%$ left
- tea 90 $\to$ 60 °C in 10 min, room 20 °C: $e^{-10k} = 4/7$; to 40 °C $\approx$ 22.4 min
- Nepal, 2011–2021: $k \approx 0.0092 \to$ doubling $\ln 2/0.0092 \approx 75$ years, not $100/0.92 = 109$ (growth compounds)
  - quick rule: doubling $\approx 70 \div$ rate %
- a curve from its slope: separate; the point fixes C

Homogeneous

- $dy/dx = f/g$, f and g homogeneous of the **same** degree $\to F(y/x)$
  - x^2+3xy: degree 2; $\sqrt{x^2+y^2}$: 1; x^2+y: not homogeneous

1. $y = vx$
2. $dy/dx = v + x\,dv/dx$
3. $dv/(F(v)-v) = dx/x$: separable
4. put $v = y/x$ back

- $(x^2+y^2)\,dx = 2xy\,dy \to x^2 - y^2 = Cx$
- x easier as the unknown: $x = vy$, $dx/dy = v + y\,dv/dy$
- $\sqrt{x^2} = |x|$: work on $x > 0$ (or $y > 0$)
- torch mirror: every ray from O reflected parallel $\to$ $dx/dy = (x + \sqrt{x^2+y^2})/y \to y^2 = 2cx + c^2$, parabolas with focus O

Linear

$$y \cdot \text{IF} = \int Q \cdot \text{IF}\,dx + C, \qquad \text{IF} = e^{\int P\,dx}$$

Why the IF:

1. want $R(y'+Py) = (Ry)' = Ry' + R'y$
2. needs $R' = RP$: $\ln R = \int P\,dx$
3. no constant needed in R

- **standard form first**: y' alone (coefficient 1); P **with its sign**
  - $xy' - 2y = x^3 \to \div x$: $P = -2/x$
- simplify $e^{\ln(\ldots)}$: $e^{\int dx/x} = x$; $e^{-2\ln x} = 1/x^2$
- $y' + y/x = x^2$: IF $= x \to y = x^3/4 + C/x$
- **linear in** x: $dx/dy + Px = Q$ (P, Q in y); IF $= e^{\int P\,dy}$
  - $(x+y+1)y' = 1$: turn over $\to x = Ce^y - y - 2$
- **RC circuit**: $R\,dQ/dt + Q/C = E$
  - charging from 0: $Q = EC(1 - e^{-t/RC})$
  - discharging: $Q = Q_0 e^{-t/RC}$
  - time constant RC: 63.2% after RC, 99.3% after $5RC$
  - the constant of integration written K (C is the capacitance)

Exact

- $M\,dx + N\,dy = 0$ exact if it is du; solution $u(x,y) = C$
- total differential: $du = (\partial u/\partial x)\,dx + (\partial u/\partial y)\,dy$
- **test**: $\partial M/\partial y = \partial N/\partial x$ (order of partial differentiation doesn't matter)

Method:

$\int M\,dx$ (y constant) $+ \int$ (terms of N without x) $dy = C$

- check: $du = M\,dx + N\,dy$
- $(2xy+3)\,dx + (x^2+4y)\,dy = 0$: $2x = 2x \to x^2y + 3x + 2y^2 = C$
- groupings on sight
  - $x\,dy + y\,dx = d(xy)$
  - $x\,dx + y\,dy = d(x^2+y^2)/2$
  - $(x\,dy - y\,dx)/x^2 = d(y/x)$
- $y\,dx + (x - y^3)\,dy = 0$: linear in x or exact; same answer

COMMON MISTAKES

✗ degree of $y = xy' + \sqrt{1+(y')^2}$ read as 1
✓ clear the root: degree 2.

✗ a constant on each side when separating
✓ one constant.

✗ IF read before y' stands alone
✓ standard form first; P with its sign.

✗ $\ln|y| = x + C \to y = e^x + C$
✓ $y = Ae^x$.

✗ doubling time $= 100 \div$ rate
✓ $\ln 2/k$ (about $70 \div$ rate).

✗ exact: every term of N integrated again
✓ only the terms without x.

✗ $y = 0$ lost by dividing by y
✓ check what you divided by.

✗ $y = vx$ with $dy/dx = v$
✓ $v + x\,dv/dx$.