Unit 13 · Anti-derivatives

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Calculus (Units 12–14 together): 22 of 75 marks; 4 MCQs, 2 short and 1 long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's 2080 model paper:** 1 mark: $\int dx/(x^2+a^2)$; 3 marks: $\int dx/(2+\cos x)$; 3 marks: a "prove that" of the same half-angle kind; partial fractions and completing the square fill a 5-mark short question
- **MCQ (1 mark)**
  - $\int dx/\sqrt{a^2-x^2}$
  - the form of partial fractions
  - $\int dx/(9-x^2)$
  - a root of a quadratic below
  - $1+\cos x$ with $t=\tan(x/2)$
  - $\int 2x\,dx/((x^2+1)(x^2+2))$
  - $\int dx/(x-1)^2$
  - definite: $\tan^{-1}$, a quarter circle
  - $F(2\pi)-F(0)=0$ for a positive integrand
  - when $\tan^{-1}$, not a log
  - $\int dx/(x(x^4+1))$
- **Short and long (5 or 8 marks, in parts)**
  - complete the square: fraction, root
  - a linear top
  - $2+\cos x$; $1+3\sin^2 x$
  - partial fractions: distinct, improper, repeated
  - $\int \sqrt{x^2-4x+8}\,dx$
  - $\sin x/(\sin x+\cos x)$
  - $1/(e^x-1)^2$; $1/(5+3\sin x)$
  - quadratic factor, then a definite integral; $1/(x^4-1)$
  - prove $\int dx/(a+b\cos x)$; use it
  - $(x^2+1)/(x^4+1)$, $(x^2-1)/(x^4+1)$; then $1/(x^4+1)$

Key terms

Standard integral: a form whose anti-derivative is known by heart.

Rational function: $P(x)/Q(x)$, two polynomials.

Proper / improper: degree of P less than that of Q / not.

Partial fractions: the simple fractions a rational function was made from.

Cover-up rule: for a factor $x-a$ once, cover it and put $x=a$ in the rest.

Half-angle substitution: $t=\tan(x/2)$.

Standard integrals

$\int \frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$	$x=a\tan\theta$ **Use:** form 1
$\int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$	$\lvert x\rvert<a; x=a\sin\theta$ **Use:** form 2
$\int \frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{x}{a}$	$x>a$ **Use:** form 3
$\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$	split into two fractions **Use:** form 4
$\int \frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$	form 4 × (−1) **Use:** form 5
$\int \frac{dx}{\sqrt{x^2\pm a^2}}=\ln\left\lvert x+\sqrt{x^2\pm a^2}\right\rvert$	$(\ln(x+s))'=1/s$ **Use:** forms 6, 7

Form 8 (by parts):

$$\int \sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Forms 9, 10 (the same way):

$$\int \sqrt{x^2\pm a^2}\,dx=\frac{x}{2}\sqrt{x^2\pm a^2}\pm\frac{a^2}{2}\ln\left\lvert x+\sqrt{x^2\pm a^2}\right\rvert$$

- every one + C; check by differentiating
- **form 4 why**: $1/(x-a)-1/(x+a)=2a/(x^2-a^2)$
- **form 8 why** (by parts, the integral comes back): $u=\sqrt{a^2-x^2}$, $v=1$; write $x^2=a^2-(a^2-x^2)$; $I=\ldots-I$, solve for I
  - its meaning: a triangle $ON\cdot NP/2$ plus a sector
- $\sin^{-1}(x/a)$ or $-\cos^{-1}(x/a)$: both right (differ by $\pi/2$); same for $\tan^{-1}$ and $-\cot^{-1}$
- forms 6, 7 = $\sinh^{-1}(x/a)$, $\cosh^{-1}(x/a)$ up to a constant
- x^2 with a coefficient: take it out or put $u=3x$
  - $\int dx/(4x^2+9)=(1/6)\tan^{-1}(2x/3)$
- ellipse's area: $4(b/a)\int_0^a \sqrt{a^2-x^2}\,dx=\pi ab$

Reducing to a standard form

Complete the square:

- take out a if $a \neq 1$; add and take away (half the x coefficient)^2
- $x^2 + 4x + 13 = (x+2)^2 + 9$; $3 + 2x - x^2 = 4 - (x-1)^2$
- then $u = x + b/(2a)$, $du = dx$
- the sign picks the form
  - $u^2 + k^2$: form 1 or 6
  - $k^2 - u^2$: form 5, 2 or 8
  - $u^2 - k^2$: form 4 or 7
- no real roots: $\tan^{-1}$; real roots: a log

A linear top, $\int (px+q)/(ax^2+bx+c)\,dx$:

- $px + q = A(2ax + b) + B$ (the bottom's derivative); compare x, then constants
- $A\int f'/f = A\ln|f|$ plus a completed square
- under a root: $A\int f'/\sqrt{f} = 2A\sqrt{f}$ plus form 2, 6 or 7
- $(3x+1)/(x^2+2x+5)$: $A = 3/2$, $B = -2$
- $\ln(x^2+2x+5)$ needs no bars: always positive

Divide by x^2 for $(x^2 \pm 1)/(x^4+1)$:

- top $1 + 1/x^2$ = derivative of $x - 1/x$; bottom $(x - 1/x)^2 + 2$
- top $1 - 1/x^2$ = derivative of $x + 1/x$; bottom $(x + 1/x)^2 - 2$
- $1/(x^4+1)$: half the difference of the two

Sines and cosines below

$a\sin^2 x + b\cos^2 x$ or $a + b\cos^2 x$: $\div \cos^2 x$ top and bottom $\to \sec^2 x$ on top; $t = \tan x$

First powers, $a + b\cos x$, $a + b\sin x$: $t = \tan(x/2)$

$\sin x = \dfrac{2t}{1+t^2}$	double angle, $\div \cos^2(x/2)$ **Use:** half-angle
$\cos x = \dfrac{1-t^2}{1+t^2}$	the same way **Use:** half-angle
$dx = \dfrac{2\,dt}{1+t^2}$	$dt = \dfrac{1}{2}\sec^2(x/2)\,dx$ **Use:** half-angle

- $1 + t^2$ cancels; a quadratic in t is left
  - $5 + 4\cos x \to \int 2\,dt/(t^2+9)$
  - $3 + 5\cos x \to \int dt/(4-t^2)$ (a log)
  - $1 + \sin x + \cos x \to \int dt/(1+t)$
- a sine brings a t of the first power: complete the square in t
- $a > b > 0$:

$$\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

- **where it holds**: $\tan(x/2)$ undefined at odd multiples of π
  - a range ending at π: the limit there
  - a range crossing π: split; $\int_0^{2\pi} dx/(2+\cos x) = 2\pi/\sqrt{3}$, not 0

Rational functions

- **improper: divide first** (like $17/5 = 3 + 2/5$)
- proper: split by the factors of the bottom

Factor of $Q(x)$	Partial fractions
$x - a$	$\dfrac{A}{x-a}$
$(x-a)^2$	$\dfrac{A}{x-a} + \dfrac{B}{(x-a)^2}$
$(x-a)^3$	up to $\dfrac{C}{(x-a)^3}$
$x^2 + px + q$, no real roots	$\dfrac{Ax+B}{x^2+px+q}$

- as many unknowns as the degree of Q
- $\times Q(x)$; **put in the roots** (cover-up), then **compare coefficients** for the rest
- $(x+5)/((x-1)(x+2))$: $A = 2$, $B = -1$
- repeated: $\int dx/(x-1)^2 = -1/(x-1)$, a power, not a log
- quadratic factor: $Ax + B$ on top $\to$ a log and a $\tan^{-1}$
- only x^2 in sight: treat x^2 as one letter
- substitutions that make a rational function
  - e^x everywhere: $t = e^x$
  - $\cos x\,dx$ with sines: $t = \sin x$
  - $\sqrt{ax+b}$: $t = \sqrt{ax+b}$
- dip chart: segment of a circle by form 8; rod at 25% of the depth $\to$ 19.6% of the fuel

COMMON MISTAKES

✗ the $+ C$ left out
✓ every indefinite integral has one.

✗ $\int dx/(4x^2+9) = (1/3)\tan^{-1}(x/3)$
✓ take the 4 out: $(1/6)\tan^{-1}(2x/3)$.

✗ partial fractions of an improper fraction
✓ divide first.

✗ $(x-a)^2$ given one term $B/(x-a)^2$
✓ also $A/(x-a)$.

✗ $x^2 + 1$ given a constant top
✓ $Ax + B$.

✗ $\int dx/(x-1)^2 = \ln|x-1|^2$
✓ $-1/(x-1)$.

✗ $F(2\pi) - F(0)$ across $x = \pi$ with $t = \tan(x/2)$
✓ split at π.

✗ $\sin^{-1}$ and $-\cos^{-1}$ answers called different
✓ they differ by a constant: both right.