Unit 12 · Derivatives

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Calculus (Units 12–14 together): 22 of 75 marks; 4 MCQs, 2 short and 1 long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's model paper:** an MCQ on a hyperbolic or inverse hyperbolic derivative, a limit by L'Hospital's rule, a tangent or a rate; 1-mark parts asking a rule ($(\sinh x)'$; f/g and f'/g'); a part on tangents and normals or a rate ("illustrate the derivative as a rate measure")
- **MCQ (1 mark)**
  - $(\cosh x)'$
  - $\cosh^2 x - \sinh^2 x$
  - $(\sinh^{-1}(x/2))'$
  - a 0/0 limit
  - an ∞/∞ limit
  - dy for $y = x^2$
  - slope of a normal
  - tangent to $y = \ln x$
  - a circle's area rate
  - a for a finite limit
  - where a tangent meets again
- **Short and long (5 or 8 marks, in parts)**
  - $(\tanh x)'$ from the definitions
  - a derivative of $\cosh^{-1}$ of a root
  - $(1 + x^2)y'' + xy' = 0$
  - limits by L'Hospital
  - approximate $\sqrt[3]{127}$, $\cos 61°$
  - error in a tile's area
  - tangent and normal, parametric
  - tangents parallel to a line
  - a sliding ladder
  - $\tanh^{-1}$ as a log; its derivative two ways
  - a stone's speed; a balloon; a shadow

Key terms

Hyperbolic cosine, sine: $\cosh x = (e^x + e^{-x})/2$, $\sinh x = (e^x - e^{-x})/2$; the even and odd parts of e^x.

Indeterminate form: 0/0 or ∞/∞ (and five more), with no value until worked out.

Differentials: $dx = \Delta x$; $dy = f'(x)\,dx$, the rise along the tangent.

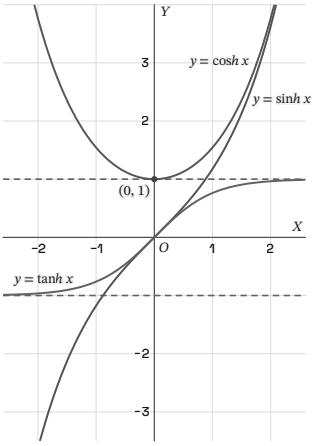
Absolute, relative, percentage error: δx; $\delta x/x$; $(\delta x/x) \times 100$.

Tangent / normal: the line through P with slope dy/dx there / at right angles to it.

Rate of change: dy/dt, how fast y changes with time.

Hyperbolic functions

- $\cosh x + \sinh x = e^x$; $\cosh x - \sinh x = e^{-x}$
- $\tanh x = \sinh x/\cosh x$; coth, sech, cosech as in trigonometry
- cosh even, a U never below 1; sinh odd, rises everywhere; tanh between −1 and 1



cosh even, never below 1; sinh odd; tanh between −1 and 1.

- **why "hyperbolic"**: $(\cosh t, \sinh t)$ lies on $x^2 - y^2 = 1$

$$\cosh^2 x - \sinh^2 x = 1$$

- square each: $(e^{2t} \pm 2 + e^{-2t})/4$; subtract → 4/4
- ÷ $\cosh^2$: $1 - \tanh^2 x = \operatorname{sech}^2 x$; ÷ $\sinh^2$: $\coth^2 x - 1 = \operatorname{cosech}^2 x$

- $\sinh 2x = 2\sinh x\cosh x$;
  $\cosh 2x = \cosh^2 x + \sinh^2 x$

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$ (**no minus**)
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{cosech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x\tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x\coth x$

- $(\tanh x)'$: quotient rule; the top is $\cosh^2 - \sinh^2 = 1 \to 1/\cosh^2 x$
- signs differ from trigonometry only for $(\cosh x)'$ and $(\operatorname{sech} x)'$
- $(\ln\cosh x)' = \tanh x$;
  $(\sinh(x^2+1))' = 2x\cosh(x^2+1)$

Inverse hyperbolic functions

$\sinh^{-1} x$ as a log:

1. $x = \sinh y = (e^y - e^{-y})/2$
2. $\times\, 2e^y$: $e^{2y} - 2xe^y - 1 = 0$, a quadratic in e^y
3. $e^y = x + \sqrt{x^2+1}$ ($e^y > 0$: only the + sign)
4. take logs

Function	As a logarithm	For
$\sinh^{-1} x$	$\ln(x + \sqrt{x^2+1})$	every x
$\cosh^{-1} x$	$\ln(x + \sqrt{x^2-1})$	$x \geq 1$
$\tanh^{-1} x$	$\frac{1}{2}\ln\frac{1+x}{1-x}$	$-1 < x < 1$
$\coth^{-1} x$	$\frac{1}{2}\ln\frac{x+1}{x-1}$	$\lvert x\rvert > 1$

- cosh is even: $\cosh^{-1} x$ takes $y \geq 0$
- $\operatorname{sech}^{-1} x = \cosh^{-1}(1/x)$; $\operatorname{cosech}^{-1} x = \sinh^{-1}(1/x)$

$$(\sinh^{-1} x)' = \frac{1}{\sqrt{x^2+1}}$$ all x. **Use:** $dx/dy = \cosh y = \sqrt{1+x^2}$

$$(\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}}$$ $x > 1$. **Use:** $\sinh y = +\sqrt{x^2-1}$

$$(\tanh^{-1} x)' = \frac{1}{1-x^2}$$ $\lvert x\rvert < 1$. **Use:** $\operatorname{sech}^2 y = 1 - x^2$

$$(\coth^{-1} x)' = \frac{1}{1-x^2}$$ $\lvert x\rvert > 1$. **Use:** $-\operatorname{cosech}^2 y = 1 - x^2$

- $(\operatorname{sech}^{-1} x)' = -1/(x\sqrt{1-x^2})$, $0 < x < 1$;
  $(\operatorname{cosech}^{-1} x)' = -1/(\lvert x\rvert\sqrt{1+x^2})$
- method: $dy/dx = 1/(dx/dy)$, then $\cosh^2 - \sinh^2 = 1$
- check by the log form: the same answer
- $\tanh^{-1}$ and $\coth^{-1}$: one derivative, never the same x: **say the domain**
- $\tanh^{-1}(\sin x) \to \sec x$; $x\sinh^{-1} x - \sqrt{1+x^2} \to \sinh^{-1} x$

L'Hospital's rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

- only for **0/0 or ∞/∞**; f, g differentiable near a, $g' \neq 0$; the right side must exist
- also for $x \to a^{\pm}$, $x \to \pm\infty$
- **why** ($f(a) = g(a) = 0$):
  $f/g = (f(x) - f(a))/(g(x) - g(a))$; $\div\,(x-a)$ top and bottom $\to f'(a)/g'(a)$
- still 0/0 or ∞/∞: use it again; **check the form every time**
- top and bottom differentiated **separately** (not the quotient rule)
- results
  - $\lim_{x\to 1}\ln x/(x-1) = 1$
  - $(e^x - 1 - x)/x^2 \to 1/2$ (twice)
  - $(x - \sin x)/x^3 \to 1/6$ (three times)
  - $\ln x/x \to 0$; $x^3/e^x \to 0$ as $x \to \infty$
- $(1-\cos x)/(x^2+x)$: once $\to 0/1 = 0$; again would wrongly give 1/2
- $(x + \sin x)/x$: $1 + \cos x$ keeps swinging; the rule says nothing

Differentials

- $\Delta y = f(x+\Delta x) - f(x)$: along the curve;
  $dy = f'(x)\,dx$: along the tangent
- small dx: $\Delta y \approx dy$; $f(x+dx) \approx f(x) + f'(x)\,dx$
- $y = x^2$: error $\Delta y - dy = (dx)^2$
- $\sqrt{26} \approx 5 + 1/10 = 5.1$
- $\sin 46°$: $dx = 1° = \pi/180 \approx 0.01745$ **radians** $\to$ 0.7194
- error in y: $dy = f'(x)\,\delta x$
- **power**: $y = kx^n \to dy/y = n\,dx/x$: % error $\times\, n$
  - 1% in a radius: 2% in an area, 3% in a volume
  - ball, $r = 10 \pm 0.05$ cm: $dV = 4\pi r^2\,dr = 20\pi \approx 62.8$ cm^3; 1.5%

Tangent and normal

$y - y_1 = m(x - x_1)$	$m = dy/dx$ at P **Use:** tangent
$(x - x_1) + m(y - y_1) = 0$	slope $-1/m$ **Use:** normal

- check the point is on the curve
- $m = 0$: tangent $y = y_1$, normal $x = x_1$; m without bound: the other way round
- implicit: differentiate both sides; parametric: $(dy/d\theta)/(dx/d\theta)$
- $y = x^3 - 2x^2 + 4$ at $(2, 4)$: $m = 4$; tangent $y = 4x - 4$, normal $x + 4y = 18$
- $x^3 + y^3 = 9$ at $(1, 2)$: $m = -x^2/y^2 = -1/4$
- $x = a\cos^3\theta$, $y = a\sin^3\theta$: $m = -\tan\theta$; the tangent cut off by the axes has length a
- parallel to a line: equal slopes; perpendicular: product -1
- light and a curved mirror: angles measured from the **normal**

Rates

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

1. name the changing quantities; draw
2. an equation true at **every** instant
3. differentiate with respect to t
4. **only now** put in the values

- ripple: $A = \pi r^2$, $dA/dt = 2\pi r\, dr/dt$; $r = 4$, 0.5 m/s $\to$ $4\pi \approx 12.6$ m^2/s
- funnel, 20 cm deep, 10 cm across: $r = h/4$, $V = \pi h^3/48$; 12 cm^3/s at $h = 8 \to 3/\pi \approx 0.95$ cm/s
- similar triangles tie a cone's or a shadow's lengths
- ladder: $x^2 + y^2 = 25 \to x\,dx/dt + y\,dy/dt = 0$
- velocity ds/dt; acceleration dv/dt; the sign says the direction
- units: the quantity's unit per unit of time
- census: $dP/dt = 0.0092P \approx 2.68$ lakh a year in 2021; dP over 5 years $\to \approx 30.5$ million in 2026

COMMON MISTAKES

✗ $(\cosh x)' = -\sinh x$
✓ $+\sinh x$: no minus.

✗ L'Hospital by the quotient rule
✓ top and bottom differentiated separately.

✗ L'Hospital on a form that isn't 0/0 or ∞/∞
✓ check the form before each step.

✗ $\sin 46^\circ$ with $dx = 1$
✓ $dx = \pi/180$: radians.

✗ the value put in before differentiating ($A = 16\pi$, rate 0)
✓ differentiate first, then substitute.

✗ normal slope $= -m$
✓ $-1/m$.

✗ $(\tanh^{-1} x)' = 1/(1 - x^2)$ for all x
✓ only $|x| < 1$.

✗ $e^y = x - \sqrt{x^2 + 1}$ kept
✓ it is negative; only $+$.