Unit 9 · Product of Vectors

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Vectors: 6 of 75 marks; with Trigonometry and Analytic Geometry, 3 MCQs (at least one on vectors), 2 short and 1 long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's model paper:** 1 mark: $|\vec{a} \times \vec{b}|$ from $\vec{a} \cdot \vec{b}$ and the two lengths; 2 marks: an identity about any four points, by the scalar product; the unit vector perpendicular to two given vectors
- **MCQ (1 mark)**
  - $\vec{i} \times \vec{k}$
  - $\vec{a} \times \vec{b} = \vec{0}$: parallel
  - $\vec{a} \cdot \vec{b}$ from lengths and angle
  - a projection
  - $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$
  - $|\vec{a} \times \vec{b}|$ from $\vec{a} \cdot \vec{b}$
  - an angle; a triangle's area
  - work done by a force
  - $\vec{a} + \vec{b} + \vec{c} = \vec{0}$: an angle
- **Short and long (5 or 8 marks, in parts)**
  - angle; projection and vector projection
  - unit vectors perpendicular to both; $\sin\theta$ and θ
  - projection law, $\sin(A + B)$ by vectors
  - area of a triangle; a perpendicular's length
  - $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$
  - collinear points: λ
  - a right-angled triangle: angle, area, normal
  - perpendicular bisectors meet; diagonals of a parallelogram

Key terms

Angle between two vectors: drawn from one point, $0 \le \theta \le \pi$.

Scalar (dot) product: $a \cdot b = |a||b|\cos\theta$, a number.

Vector (cross) product: $|a||b|\sin\theta\,\hat{n}$, a vector perpendicular to both.

Right-hand rule: fingers from a to b; the thumb along $\hat{n}$ (a screw's way).

Projection of b on a: how far b reaches along a; negative behind.

Work: $W = F \cdot s$; **torque**: $\tau = r \times F$.

Scalar product

$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$	components **Use:** dot product
$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{\lvert\vec{a}\rvert\lvert\vec{b}\rvert}$	one angle in $[0, \pi]$ **Use:** angle
$\dfrac{\vec{a} \cdot \vec{b}}{\lvert\vec{a}\rvert} = \vec{b} \cdot \hat{a}$	 **Use:** projection of $\vec{b}$ on $\vec{a}$
$\dfrac{\vec{a} \cdot \vec{b}}{\lvert\vec{a}\rvert^2}\vec{a}$	along $\vec{a}$ **Use:** vector projection

Lengths ($\vec{a} \cdot \vec{a} = |\vec{a}|^2$):

$$|\vec{a} \pm \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \pm 2\vec{a} \cdot \vec{b}$$

- $\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$; $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$
- **why the components**: $\vec{a} \cdot \vec{b} = (|\vec{a}|^2 + |\vec{b}|^2 - |\vec{b} - \vec{a}|^2)/2$ (Pythagoras), then put in components
- sign: **positive acute; zero right angle; negative obtuse**
- $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = \mathbf{0}$ (non-zero vectors)
- properties: commutative, distributive, numbers come out
  - $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$
  - $|\vec{a} \cdot \vec{b}| \le |\vec{a}||\vec{b}|$
- unit vectors (l, m, n): $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$
- work: $\vec{F} = (3, 2, -1)$ from $(1, -1, 2)$ to $(4, 3, 1)$: $\vec{s} = (3, 4, -1)$, $W = 18$ J
- $(1, 2, 2)$, $(2, -1, 2)$: $\cos\theta = 4/9$, $\theta \approx 63.6°$

Proofs by the scalar product

- perpendicular: a product is 0; a length: a vector times itself
- **cosine law**: $\vec{a} = \overrightarrow{CB}$, $\vec{b} = \overrightarrow{CA}$; $c^2 = |\vec{a} - \vec{b}|^2 = a^2 + b^2 - 2ab\cos C$

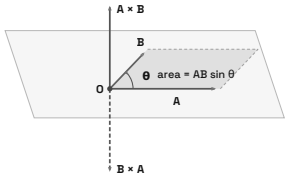
- **cos $(A - B)$**: unit vectors $(\cos A, \sin A)$, $(\cos B, \sin B)$; the angle between is $A - B$
  - put $-B$ for B: $\cos(A + B)$
- **rhombus**: $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = 0$: diagonals $\perp$
- any four points: write each vector as end minus start; everything cancels
- parallelogram: $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2|\vec{a}|^2 + 2|\vec{b}|^2$

Vector product

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$$

	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{a}$	a_1	a_2	a_3
$\vec{b}$	b_1	b_2	b_3

- as a determinant with $\vec{i}, \vec{j}, \vec{k}$ in the first row; the $\vec{j}$ term with a **minus**
- $\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$: round the cycle plus, backwards minus
- $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$



$\vec{A} \times \vec{B}$ at right angles to both; $\vec{B} \times \vec{A}$ the other way.

$\sin\theta = \dfrac{\lvert\vec{a} \times \vec{b}\rvert}{\lvert\vec{a}\rvert\lvert\vec{b}\rvert}$	the same for θ and $180° - \theta$ **Use:** sine of the angle
$\hat{n} = \pm\dfrac{\vec{a} \times \vec{b}}{\lvert\vec{a} \times \vec{b}\rvert}$	two of them **Use:** unit vectors $\perp$ both
$\lvert\vec{a} \times \vec{b}\rvert$	base × height **Use:** parallelogram's area
$\frac{1}{2}\lvert\overrightarrow{AB} \times \overrightarrow{AC}\rvert$	from three points **Use:** triangle's area
$\lvert\vec{a} \times \vec{b}\rvert^2 + (\vec{a} \cdot \vec{b})^2 = \lvert\vec{a}\rvert^2\lvert\vec{b}\rvert^2$	$\sin^2 + \cos^2 = 1$ **Use:** both products

- **why the components**: $\vec{c} \cdot \vec{a} = \vec{c} \cdot \vec{b} = 0$ ($\perp$ both); Lagrange's identity gives the length $|\vec{a}||\vec{b}|\sin\theta$
- **check**: dot the answer with $\vec{a}$ and with $\vec{b}$: both 0
- $(1, 2, -1) \times (3, -1, 2) = (3, -5, -7)$; $\hat{n} = \pm(3, -5, -7)/\sqrt{83}$
- $|\vec{a}| = 5, |\vec{b}| = 4, \vec{a} \cdot \vec{b} = 12$: $|\vec{a} \times \vec{b}| = \sqrt{400 - 144} = 16$; triangle 8
- in the plane: $(a_1 b_2 - a_2 b_1)\vec{k}$; positive for an anticlockwise turn
- the vector product is **an area with a direction** (normal to the plane)

Properties of the vector product

- **anticommutative**: $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$
- $\vec{a} \times \vec{b} = \vec{0} \iff \vec{a} \parallel \vec{b}$ (non-zero)
- **collinear** A, B, C: $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{0}$
- distributive; numbers come out
- **not associative**: $(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0}, \vec{i} \times (\vec{i} \times \vec{j}) = -\vec{j}$
- $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$: the diagonals' parallelogram is twice the area

Proofs by the vector product

- **sine law**: $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ round the triangle
  1. cross with $\vec{a}$, then $\vec{b}$: $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$
  2. lengths: $ab \sin C = bc \sin A = ca \sin B$ (the angle between is $180° - C$)
  3. divide by abc
- the same line: $\Delta = ab \sin C/2$
- **sin $(A - B)$**: $\overrightarrow{OQ} \times \overrightarrow{OP} = \sin(A - B)\vec{k} = (\sin A \cos B - \cos A \sin B)\vec{k}$
- **a plot from its corners** (surveyor's or shoelace formula): area $= |\Sigma (x_r y_{r+1} - x_{r+1} y_r)|/2$, corners in order
  - triangles from one corner; each a half vector product
  - five corners: $216 + 292 + 99 = 607$ m$^2 \approx 1$ ropani 3 aana (1 ropani ≈ 508.7 m^2)

COMMON MISTAKES

✗ the angle by $\sin^{-1}$ of $|a \times b| \div |a||b|$
✓ it can't tell θ from $180° - \theta$: use the dot product ($80.4°$ was really $99.6°$).

✗ $b \times a = a \times b$
✓ $-a \times b$.

✗ the j term with a plus
✓ minus: $(a_3 b_1 - a_1 b_3)$.

✗ one unit vector perpendicular to both
✓ two: $\pm$.

✗ the triangle's area $= |AB \times AC|$
✓ half of it.

✗ $a \times b = a \times c \to b = c$
✓ only $b - c$ parallel to a.

✗ $AB = a - b$
✓ end minus start: $b - a$.