# Unit 8 · Conic Section

STUDY NOTES

**WHAT NEB'S GRADE 12 EXAM ASKS**

- **CDC's grid (2078):** Analytic Geometry: 9 of 75 marks, all from this unit; with Trigonometry and Vectors, 3 MCQs, 2 short and 1 long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's model paper:** MCQs: the length of a tangent to a circle; the tangent to a parabola at a point; short: a hyperbola whose foci are an ellipse's; a runner whose distances from two flag posts add to 10 m (illustrate and interpret, with a diagram); long: a triangle inscribed in a parabola, its area
- **MCQ (1 mark)**
  - $e$ of a parabola
  - tangent to $x^2 + y^2 = a^2$
  - asymptotes
  - latus rectum of $x^2 = -12y$
  - foci of a hyperbola
  - $c$ for tangency
  - $k$ from a tangent's length
  - the normals' common point
  - other focal distance on an ellipse
  - chord of contact
- **Short and long (5 or 8 marks, in parts)**
  - tangents of a given direction; contact points
  - show a line touches; the normal
  - vertex, focus, directrix of a shifted parabola
  - tangent and normal to $y^2 = 4ax$; where the normal meets again
  - an ellipse from vertices and latus rectum
  - a hyperbola with an ellipse's foci
  - a locus with $PS = e \cdot PM$
  - focal chord's circle touches the directrix
  - triangle in a parabola: area
  - tangents from an outside point
  - a shifted hyperbola
  - a path with fixed sum of distances
  - an elliptical lawn: pegs, rope, sprinkler
  - tangents at the latus rectum's ends

## Key terms

**Secant / tangent**: a line cutting a circle at two points / touching it at one.

**Normal**: the line through the point of contact at right angles to the tangent; for a circle, through the center.

**Length of the tangent** from $P$: $PT$, $T$ the point of contact.

**Conic**: the points with $PS = e \cdot PM$: focus $S$, directrix, eccentricity $e$.

**Ellipse** $e < 1$; **parabola** $e = 1$; **hyperbola** $e > 1$; circle $e = 0$.

**Axis**: through the focus at right angles to the directrix; the curve is symmetric about it.

**Latus rectum**: the chord through a focus at right angles to the axis.

**Focal distance**: a point's distance from a focus.

**Asymptote**: a line a hyperbola's branch closes in on without meeting.

**Rectangular hyperbola**: $a = b$, asymptotes $y = \pm x$, $e = \sqrt{2}$.

## Circle

$x^2 + y^2 + 2gx + 2fy + c = 0$: center $(-g, -f)$, $r = \sqrt{g^2 + f^2 - c}$

| | |
|---|---|
| $c^2 = a^2(1 + m^2)$ | line touches $x^2 + y^2 = a^2$<br>**Use:** tangency |
| $y = mx \pm a\sqrt{1 + m^2}$ | two, one each side<br>**Use:** slope $m$ |
| $xx_1 + yy_1 = a^2$ | $(x_1, y_1)$ **on** it<br>**Use:** tangent there |
| $PT = \sqrt{S_1}$ | $S_1$: $(x_1, y_1)$ put in $S$<br>**Use:** tangent's length |

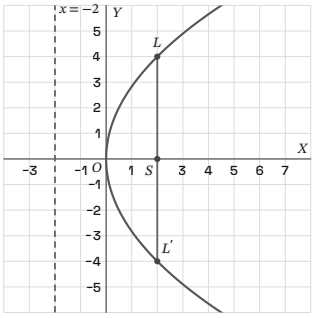
**Tangent at $(x_1, y_1)$, general circle:**

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

- $S_1 = {x_1}^2 + {y_1}^2 + 2gx_1 + 2fy_1 + c$
- line and circle: perpendicular from the center $p$
  - $p < r$: cuts
  - $p = r$: touches
  - $p > r$: misses
- **any circle: a line touches when its distance from the center = the radius**
- tangency by algebra: put the line into the circle; discriminant 0 (equal roots)
- $y = mx + c$ touches $x^2 + y^2 = a^2$ at $(-a^2m/c, a^2/c)$
- tangent $\perp$ radius; normal = the line through the point and the center
- **rule**: $xx_1$ for $x^2$, $yy_1$ for $y^2$, $x + x_1$ for $2x$, $y + y_1$ for $2y$
- tangent parallel to $3x + 4y = 0$: $3x + 4y + k = 0$, distance $= r \rightarrow$ two values of $k$
- $S_1 > 0$ outside; $= 0$ on; $< 0$ inside (no tangent)
- $PT^2 = CP^2 - r^2$ (right angle at $T$)
- **tangents from an outside point**: line of slope $m$ through $P$; distance from center $= r \rightarrow$ quadratic in $m$
  - one root only: the other tangent is upright, $x = x_1$
- from a point outside, $xx_1 + yy_1 = a^2$ is the **chord of contact**, not a tangent

## Parabola

**Standard equation:** $S(a, 0)$, directrix $x = -a$; $PS = PM$: $(x - a)^2 + y^2 = (x + a)^2 \rightarrow y^2 = 4ax$




$y^2 = 8x$: focus $S(2, 0)$, directrix $x = -2$, latus rectum 8.

| Equation | Opens | Focus | Directrix | Axis |
|---|---|---|---|---|
| $y^2 = 4ax$ | right | $(a, 0)$ | $x = -a$ | $y = 0$ |
| $y^2 = -4ax$ | left | $(-a, 0)$ | $x = a$ | $y = 0$ |
| $x^2 = 4ay$ | up | $(0, a)$ | $y = -a$ | $x = 0$ |
| $x^2 = -4ay$ | down | $(0, -a)$ | $y = a$ | $x = 0$ |

- latus rectum $4a$ (ends $(a, \pm 2a)$); focal distance $x_1 + a$
- vertex at $(h, k)$: $(y - k)^2 = 4a(x - h)$; focus $(h + a, k)$, directrix $x = h - a$
  - complete the square; the vertex is halfway between focus and directrix
- from a focus and directrix: $PS^2 = PM^2$
- parameter: $(at^2, 2at)$; tangent there $ty = x + at^2$

| | |
|---|---|
| $yy_1 = 2a(x + x_1)$ | slope $2a/y_1$<br>**Use:** tangent at $(x_1, y_1)$ |
| $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ | slope $-y_1/(2a)$<br>**Use:** normal |

- tangent by implicit differentiation: $2y\,dy/dx = 4a$
- $y^2 = 8x$ at $(2, 4)$: tangent $x - y + 2 = 0$, normal $x + y - 6 = 0$
- **reflection**: tangent at $P$ meets the axis at $T(-x_1, 0)$; $ST = SP = x_1 + a \rightarrow$ isosceles $\rightarrow$ equal angles
  - rays parallel to the axis reflect through the focus: dish antenna, solar cooker, torch
- triangle on $y^2 = 4ax$ at ordinates $y_1, y_2, y_3$: points $(y^2/(4a), y)$; area $= |(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|/(8a)$

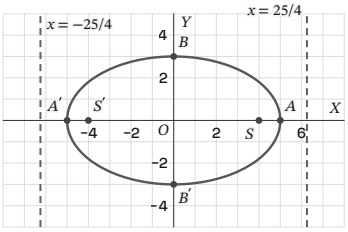
## Ellipse

**Standard equation:** $S(ae, 0)$, directrix $x = a/e$, $e < 1$

1. $(x - ae)^2 + y^2 = e^2(x - a/e)^2 = (ex - a)^2$
2. the $-2aex$ cancels: $x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

3. divide; $b^2 = a^2(1 - e^2)$

| | |
|---|---|
| $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ | $a > b$, $b^2 = a^2(1 - e^2)$<br>**Use:** ellipse |
| $e = \sqrt{1 - b^2/a^2}$ | $ae = \sqrt{a^2 - b^2}$<br>**Use:** eccentricity |
| $S(\pm ae, 0)$ | directrices $x = \pm a/e$<br>**Use:** foci |
| $\frac{2b^2}{a}$ | at $x = ae$, $y = \pm b^2/a$<br>**Use:** latus rectum |
| $PS + PS' = 2a$ | $PS = a - ex$, $PS' = a + ex$<br>**Use:** focal distances |



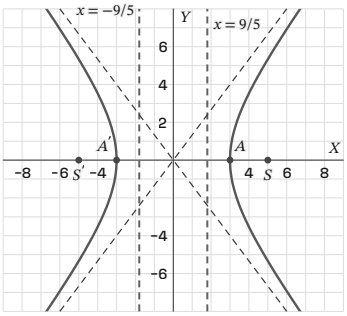

$x^2/25 + y^2/9 = 1$: $e = 4/5$, foci $(\pm 4, 0)$, directrices $x = \pm 25/4$.

- vertices $(\pm a, 0)$; major axis $2a$, minor $2b$
- $b > a$: major axis along the Y-axis; foci $(0, \pm be)$, $e = \sqrt{1 - a^2/b^2}$, latus rectum $2a^2/b$
  - always look which denominator is larger
- center $(h, k)$: complete the squares; everything moves with the center; $e$, axes, latus rectum don't
- **string (pegs) construction**: pegs at the foci, rope $2a$
  - bed 10 m × 8 m: pegs 3 m from the center, rope 10 m
  - runner, sum 26 m, posts 24 m apart: $2a = 26$, $2ae = 24$, $b = 5$: $x^2/169 + y^2/25 = 1$
- nearest and farthest from a focus: $a - ae$, $a + ae$
- Earth's orbit: $e = 0.0167$; Sun $ae \approx 2.5$ million km from the center; flattening goes with $e^2$

## Hyperbola

Same steps with $e > 1$; $1 - e^2 < 0$, so $b^2 = a^2(e^2 - 1)$

| | |
|---|---|
| $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ | $b^2 = a^2(e^2 - 1)$<br>**Use:** hyperbola |
| $e = \sqrt{1 + b^2/a^2}$ | $ae = \sqrt{a^2 + b^2}$<br>**Use:** eccentricity |
| $y = \pm\frac{b}{a}x$ | branches close in<br>**Use:** asymptotes |
| $\lvert PS - PS' \rvert = 2a$ | $PS = ex - a$, $PS' = ex + a$<br>**Use:** focal distances |




$x^2/9 - y^2/16 = 1$: $e = 5/3$, foci $(\pm 5, 0)$, asymptotes $y = \pm 4x/3$.

- **two branches**: no point with $-a < x < a$
- vertices $(\pm a, 0)$; transverse axis $2a$, conjugate $2b$; latus rectum $2b^2/a$
- Y-axis form $y^2/a^2 - x^2/b^2 = 1$: vertices $(0, \pm a)$, foci $(0, \pm ae)$
- $a$ is under the **positive** square; $b$ may be larger
- focus's distance $c = ae$: ellipse $c^2 = a^2 - b^2$; hyperbola $c^2 = a^2 + b^2$
- hyperbola with an ellipse's foci: find $c$ from the ellipse, then $a = c/e$, $b^2 = c^2 - a^2$
- **LORAN**: time difference → difference of distances $2a$ → one branch; two pairs of stations → two curves cross at the ship

### COMMON MISTAKES

✗ $xx_1 + yy_1 = a^2$ from a point outside, as a tangent
✓ only at a point **on** the circle; outside it is the chord of contact.

✗ $x^2/16 + y^2/25 = 1$: foci on the X-axis
✓ the larger denominator is under $y$: foci $(0, \pm 3)$.

✗ hyperbola: $b^2 = a^2(1 - e^2)$
✓ $a^2(e^2 - 1)$; $c^2 = a^2 + b^2$.

✗ ellipse: $c^2 = a^2 + b^2$
✓ $a^2 - b^2$.

✗ $x^2 = -12y$: latus rectum $-12$
✓ a length: 12.

✗ the vertex of $(y - 2)^2 = -8(x - 1)$ at $O$
✓ at $(1, 2)$.

✗ a missing tangent from $P$ when the quadratic in $m$ has one root
✓ it is upright, $x = x_1$.