Unit 6 · Properties of a Triangle

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Trigonometry (Units 6–7 together): 6 of 75 marks; with Analytic Geometry and Vectors, 3 MCQs (at least one on trigonometry), 2 short and 1 long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's model paper:** MCQ: what kind of triangle has two given sides and the angle between them; 2 marks: prove an identity with $\cos^2(A/2)$ and s; 2 marks: prove an identity with $b\cos C$ and $c\cos B$
- **MCQ (1 mark)**
  - cos C in sides
  - $b\cos C + c\cos B$
  - a side from two sides and the angle
  - the largest angle from $\sin A : \sin B : \sin C$
  - C from $c^2 = a^2 + ab + b^2$
  - $\tan(A/2)$; an area from three sides
  - clock hands' tips
  - $\tan((B-C)/2)$
  - kind of triangle from a condition
- **Short and long (5 or 8 marks, in parts)**
  - identities: change sides to $k\sin A$…, or use projection
  - two angles by the tangent law; check by the cosine law
  - largest angle twice the smallest (4, 5, 6)
  - cosine law from the projection laws
  - $a\cos^2(C/2) + c\cos^2(A/2) = s$
  - prove a half-angle law; area, $\tan(A/2)\tan(B/2) = (s-c)/s$

Key terms

Notation: angles A, B, C; opposite sides $a = BC$, $b = CA$, $c = AB$; $A + B + C = 180°$.

Semi-perimeter $s = (a+b+c)/2$.

Projection of a side on another: its shadow, $c\cos B$ of AB on BC.

Heron's formula: the area from three sides.

Formulas

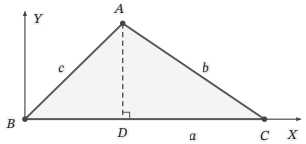
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$	$a = k\sin A$, … **Use:** sine law
$a^2 = b^2 + c^2 - 2bc\cos A$	$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ **Use:** cosine law
$\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$	two sides, angle between **Use:** tangent law
$a = b\cos C + c\cos B$	shadows of the other two **Use:** projection law
$\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$	$s = \dfrac{a+b+c}{2}$ **Use:** half-angle
$\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$	positive root **Use:** half-angle
$\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$	divide the two **Use:** half-angle
$\Delta = \dfrac{1}{2}bc\sin A$	two sides, angle between **Use:** area
$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$	three sides **Use:** Heron

Needed: $\sin(B+C) = \sin A$, $\cos(B+C) = -\cos A$; $\cos A = 1 - 2\sin^2(A/2) = 2\cos^2(A/2) - 1$; sum to product

One line behind every law

B at the origin, C at $(a, 0)$: A seen from B and from C:

$$A = (c\cos B, c\sin B) = (a - b\cos C, b\sin C)$$



The height AD is $c\sin B$ from B and $b\sin C$ from C.

- true for every triangle (obtuse B: $c\cos B < 0$)

- y's equal → sine law; x's equal → projection law; distance CA → cosine law

Sine law

1. $c \sin B = b \sin C$ (both the height AD)
2. divide by $\sin B \sin C$: $c/\sin C = b/\sin B$
3. another corner at the origin: $a/\sin A = c/\sin C$

- each side $\propto$ the sine of the angle opposite; **longest side faces the largest angle**
- area: base $a \times$ height $c \sin B$, halved
- in identities: put $a = k \sin A$ everywhere; then $\sin A = \sin(B + C)$
  - $\sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y$
- **distance to a peak**: base 20 km, angles 84° and 87° → angle at peak 9°; $S_1P = 20 \sin 87°/\sin 9° \approx 127.7$ km
  - small angle at the peak: a 0.1° error moves the answer about 1.4 km

Cosine law

1. b = distance from $C(a, 0)$ to $A(c \cos B, c \sin B)$
2. $b^2 = (c \cos B - a)^2 + c^2 \sin^2 B = c^2 + a^2 - 2ca \cos B$

- for three sides, or two sides and the angle between them (the sine law can't start)
- $A = 90°$: Pythagoras
- $a^2 < b^2 + c^2$: A acute; =: right; >: obtuse
- sides 3, 5, 7: $\cos C = -1/2$, $C = 120°$
- $c^2 = a^2 + ab + b^2 \to \cos C = -1/2$: 120°

Tangent law

1. sine law: $(b - c)/(b + c) = (\sin B - \sin C)/(\sin B + \sin C)$
2. sum to product: $= \cot \dfrac{B + C}{2} \tan \dfrac{B - C}{2}$
3. $(B + C)/2 = 90° - A/2$: $\cot \dfrac{B + C}{2} = \tan \dfrac{A}{2}$

- gives $B - C$; with $B + C = 180° - A$, both angles, no third side
- $b = \sqrt{3} + 1$, $c = \sqrt{3} - 1$, $A = 60° \to B - C = 90°$ $\to B = 105°$, $C = 15°$

Projection laws

$$a = b \cos C + c \cos B$$
$$b = c \cos A + a \cos C$$
$$c = a \cos B + b \cos A$$

- from the two x's of A: $c \cos B = a - b \cos C$
- each side = the sum of the projections of the other two on it
- obtuse B: D beyond B, $c \cos B$ negative, still holds
- **cosine law from them**: $\times a$, $\times b$, $\times(-c)$, add → $a^2 + b^2 - c^2 = 2ab \cos C$
- in identities: look for pairs $b \cos C + c \cos B\ (= a)$
  - $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$

Half-angle laws

- $b + c - a = 2(s - a)$, $a - b + c = 2(s - b)$, $a + b - c = 2(s - c)$

1. $2\sin^2(A/2) = 1 - \cos A$; cosine law → $(a^2 - (b - c)^2)/(2bc)$
2. difference of squares → $2(s - b) \cdot 2(s - c)/(2bc)$
3. $2\cos^2(A/2) = 1 + \cos A$
   - $= ((b + c)^2 - a^2)/(2bc)$
   - $= 2s \cdot 2(s - a)/(2bc)$
4. $A/2 < 90°$: positive roots; divide for tan

- **Heron**: $\Delta = bc \sin(A/2) \cos(A/2) \to \sqrt{s(s - a)(s - b)(s - c)}$; $\sin A = 2\Delta/(bc)$
- 13, 14, 15: $s = 21$, $\Delta = 84$, $\tan(A/2) = 1/2$, $\sin A = 4/5$
- $\tan(A/2) \tan(B/2) = (s - c)/s$

Kind of triangle from a condition

- $a \cos A = b \cos B$: $\sin 2A = \sin 2B \to A = B$ or $A + B = 90°$: isosceles or right-angled
- $\cos A = \sin B/(2 \sin C)$: with $\sin B = \sin(A + C)$ $\to \sin(A - C) = 0$: $A = C$, isosceles
- two sides and the angle between them: cosine law for the third, then compare

COMMON MISTAKES

✗ $a/\sin A$ with a not opposite A
✓ each side over the sine of the angle **opposite** it.

✗ $\cos A = (a^2 + b^2 - c^2)/(2ab)$
✓ $(b^2 + c^2 - a^2)/(2bc)$: the side opposite A is subtracted.

✗ a negative cos rejected
✓ it means an obtuse angle (120°).

✗ $\sin(A/2) = -\sqrt{\ldots}$
✓ $A/2 < 90°$: positive root.

✗ $s = a + b + c$
✓ half of it.

✗ $a \cos A = b \cos B \to$ isosceles only
✓ or right-angled ($A + B = 90°$).