Unit 5 · Matrix Based System of Linear Equations

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Algebra (Units 1–5 together): 20 of 75 marks; 2 MCQs, 2 short and 1 long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's model paper:** a 3-mark part: solve three equations by the row-equivalent method
- **MCQ (1 mark)**
  - how D_y is formed
  - which is not a row operation
  - parallel lines: no solution
  - $|A| \neq 0$: exactly one solution
  - D of a 2×2 system; x by Cramer
  - read the case from a reduced matrix
  - $X = A^{-1}B$ from a given A^{-1}
  - a value of k or λ for no or many solutions
- **Short and long (5 or 8 marks, in parts)**
  - Cramer's rule: 2 and 3 unknowns
  - A^{-1}, then $X = A^{-1}B$ for two right sides
  - the row-equivalent method
  - a parabola through three points
  - for which k, λ, μ: one, none, infinitely many
  - Leontief's model: $X = AX + F$, $(I - A)^{-1}$

Key terms

Linear equation: each unknown to the first power only, never two unknowns multiplied.

System, solution: equations that hold together; values satisfying every one.

Coefficient matrix A, **unknowns** X, **constants** B: the system is $AX = B$.

Augmented matrix $[A \mid B]$: A with B as an extra column.

Consistent: at least one solution; **inconsistent**: none.

Row-equivalent matrices: joined by elementary row operations; same solutions.

Pivot: the diagonal entry made 1 while its column is cleared.

Formulas

$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$	$D = \lvert A \rvert \neq 0$ **Use:** Cramer's rule
$X = A^{-1}B$	$\lvert A \rvert \neq 0$; multiply on the **left** **Use:** matrix (inverse) method
$A^{-1} = \frac{1}{\lvert A \rvert}\text{adj}A$	adj = transpose of the cofactors **Use:** inverse
$[A \mid B] \sim [I \mid X]$	row operations **Use:** row-equivalent method
$[A \mid I] \sim [I \mid A^{-1}]$	the same operations **Use:** inverse by rows

- D_x: D with x**'s column replaced by the constants** (D_y, D_z the same)
- 2×2: $D = a_1b_2 - a_2b_1$, $D_x = c_1b_2 - c_2b_1$, $D_y = a_1c_2 - a_2c_1$

Three kinds of answer

Two lines (three planes)	Solutions
cross at one point	exactly one
parallel (no common point)	none
the same line (share a line)	infinitely many

- never exactly two
- every method must say **which case**, not only find a point
- an unknown missing from an equation: coefficient 0
- Kirchhoff's laws: one equation per junction and loop → $AX = B$ (a negative current flows against its arrow)

Cramer's rule

Why, two unknowns: eliminate y ($\times b_2$, $\times b_1$, subtract): $(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1$, i.e. $xD = D_x$

Why, three unknowns:

1. xD: factor x into column 1 (a_1x, a_2x, a_3x)
2. $C_1 \to C_1 + yC_2 + zC_3$: value unchanged; column 1 becomes the constants
3. so $xD = D_x$; likewise $yD = D_y$, $zD = D_z$

- each unknown on its own (one current, one price)
- $D = 0$, two unknowns: same slope, parallel or the same line
  - D_x or $D_y \neq 0$: **no solution** (0 = non-zero)

- all zero: **infinitely many** (equations multiples of each other)

- $D = 0$, three unknowns
  - any of D_x, D_y, $D_z \neq 0$: no solution
  - all zero: **can't decide** ($x + y + z = 1, 2, 3$: all zero, no solution) $\rightarrow$ row operations

The inverse method

1. $AX = B$, $|A| \neq 0$
2. multiply both sides by A^{-1} **on the left** (order matters)
3. $A^{-1}AX = IX = X$
4. $X = A^{-1}B$, the only solution

- $|A| = 0$: no inverse; none or infinitely many (decide by rows)
- worth it when one A meets many Bs: two fertilizer doses, many demands
- cofactor $C_{ij} = (-1)^{i+j} \times$ minor; adj A = transpose of the cofactor matrix
- $x + y + z = 6$, $x - y + z = 2$, $2x + y - z = 1$: $|A| = 6$, $(1, 2, 3)$
- **fertilizer**: urea 46% N; DAP 18% N, 46% P_2O_5; MoP 60% K_2O
  - $0.46u + 0.18d = N$, $0.46d = P_2O_5$, $0.60m = K_2O$
  - 100:30:30 a hectare $\rightarrow$ about 192, 65, 50 kg; a ropani ≈ 0.0509 ha
  - -0.851 in A^{-1}: more phosphate $\rightarrow$ less urea (DAP brings N too)
- **Leontief**: $X = AX + F \rightarrow (I - A)X = F \rightarrow X = (I - A)^{-1}F$

Row-equivalent method

Elementary row operations (never change the solutions; each can be undone):

1. swap two rows, $R_i \leftrightarrow R_j$
2. multiply a row by $k \neq 0$, $R_i \rightarrow kR_i$ ($k = 0$ throws an equation away)
3. add a multiple of another row, $R_i \rightarrow R_i + kR_j$

Steps:

1. write $[A \mid B]$
2. one column at a time: make the pivot 1, then every other entry of its column 0
3. reach $[I \mid X]$; read x, y, z
4. check in an equation

- the order of operations may differ; the last matrix is the same

- **last rows tell the case**
  - $(0\ 0\ 0 \mid c)$, $c \neq 0$: says $0 = c \rightarrow$ **no solution**
  - a row of zeros, no such row: **infinitely many**; one unknown free ($z = t$)
- $x + y + z = 3$, $x + 2y + 3z = 6$, $2x + 3y + 4z = 9$: row 3 = row 1 + row 2 $\rightarrow (t, 3 - 2t, t)$, a line
- same with 10 for 9: last row $(0\ 0\ 0 \mid 1) \rightarrow$ none (planes meet in pairs only)
- inverse by rows: each operation = multiplying on the left by a matrix E; $EA = I \rightarrow E = A^{-1}$
  - a row of zeros on the left: $|A| = 0$, no inverse
- conditions on λ, μ: reduce; the last row $(0\ 0\ \lambda - a \mid \mu - b)$
  - $\lambda \neq a$: exactly one
  - $\lambda = a$, $\mu \neq b$: none
  - $\lambda = a$, $\mu = b$: infinitely many

Which method?

Method	Good for	Fails when
Cramer	one unknown alone; 2 or 3 unknowns	$D = 0$ (3 unknowns: can't decide)
inverse	one A, many B	$\|A\| = 0$
row operations	every case; large systems (computers)	never: it says which case

- $AX = O$ (all constants 0): always $X = O$; $|A| = 0 \rightarrow$ infinitely many

COMMON MISTAKES

✗ D_y: replace the y row by the constants
✓ the y **column**.

✗ $X = BA^{-1}$
✓ $A^{-1}B$: multiply on the left.

✗ $D = 0$ and $D_x = D_y = D_z = 0$, three unknowns: infinitely many
✓ can't tell; reduce by rows.

✗ multiply a row by 0
✓ $k \neq 0$ only.

✗ a row of zeros means no solution
✓ $(0\ 0\ 0 \mid c)$, $c \neq 0$, means none; a zero row, many.

✗ the solution found, no check
✓ put it back into an equation.