Unit 4 · Sequence and Series

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Algebra (Units 1–5 together): 20 of 75 marks; 2 MCQs, 2 short and 1 long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's model paper:** MCQ: the sum of the first 5 odd numbers; long, 3 marks: prove by induction a sum of products $(1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots)$
- **MCQ (1 mark)**
  - $\Sigma\, k^2$'s formula
  - the first step of induction
  - sum of the first n odd numbers
  - $\Sigma\,(k^2 + 1)$, $\Sigma\, k^3$ to 10
  - the nth term of a series of products
  - $11^2 + \cdots + 20^2$
  - the bottom layer of a pyramid
  - largest n with $\Sigma\, k^2 < 1000$
- **Short and long (5 or 8 marks, in parts)**
  - sum of $1^2 + 3^2 + 5^2 + \cdots$
  - sum of $2 \cdot 1^2 + 3 \cdot 2^2 + \cdots$
  - n from $\Sigma\, k^3 = 2025$
  - prove a sum by induction (fractions, cubes)
  - $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots$: nth term, sum, then induction

Key terms

Σ notation: $\Sigma\, t_k$, $k = 1$ to $n = t_1 + t_2 + \cdots + t_n$.

Telescoping: adding differences so that all but the first and last terms cancel.

Base step: show $P(1)$ is true.

Inductive step: if $P(k)$ is true, then $P(k+1)$ is true.

Induction hypothesis: $P(k)$, **assumed** (not known) in the inductive step.

Formulas

$\Sigma\, k = \dfrac{n(n+1)}{2}$	$1 + 2 + \cdots + n$ **Use:** natural numbers
$\Sigma\, k^2 = \dfrac{n(n+1)(2n+1)}{6}$	$1^2 + \cdots + n^2$ **Use:** squares
$\Sigma\, k^3 = \left[\dfrac{n(n+1)}{2}\right]^2$	$= (\Sigma\, k)^2$ **Use:** cubes
$1 + 3 + \cdots + (2n-1) = n^2$	first n odd **Use:** odd numbers
$2 + 4 + \cdots + 2n = n(n+1)$	first n even **Use:** even numbers

Rules for Σ ($k = 1$ to n):

- $\Sigma\,(t_k + u_k) = \Sigma\, t_k + \Sigma\, u_k$
- $\Sigma\, c\, t_k = c\, \Sigma\, t_k$
- $\Sigma\, c = nc$ (n terms, each c)
- **never** over products: $\Sigma\, k^2 \neq (\Sigma\, k)^2$ ($n = 2$: $5 \neq 9$)

Summing a series

1. find the kth term t_k (each factor's own AP)
2. multiply out into powers of k
3. sum each power with the three formulas
4. factorize

- $1 \cdot 2 + 2 \cdot 3 + \cdots$: $t_k = k^2 + k \rightarrow$ $S_n = n(n+1)(n+2)/3$ (20 terms: 3080)
- $1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + \cdots$: $t_k = k(2k+1)$
- $1^2 + 3^2 + 5^2 + \cdots$: $t_k = (2k-1)^2$
- **from m to n**: sum to n − sum to $m-1$
  - $11^3 + \cdots + 20^3 = 210^2 - 55^2 = 41{,}075$
- check with $n = 1, 2, 3$
- n terms each about $k^2 \rightarrow$ sum grows like $n^3/3$

Finding $\Sigma\, k^2$ (telescoping)

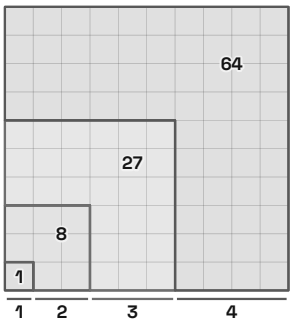
1. $(k+1)^3 - k^3 = 3k^2 + 3k + 1$
2. add for $k = 1$ to n: left side telescopes to $(n+1)^3 - 1$
3. right side: $3S + 3n(n+1)/2 + n$

4. solve for S, take out $(n+1)$:
   $3S = n(n+1)(2n+1)/2$

- **pyramid of oranges**, 10 on a side:
  $1^2 + \cdots + 10^2 = 10 \times 11 \times 21/6 = 385$

Finding $\Sigma\ k^3$

1. $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$
2. add; left telescopes:
   $(n+1)^4 - 1 = 4T + 6\Sigma\ k^2 + 4\Sigma\ k + n$
3. $4T = n^4 + 2n^3 + n^2 = n^2(n+1)^2$



A square of side $1 + 2 + 3 + 4 = 10$ cut into L's of areas 1, 8, 27, 64.

- **sum of the first n cubes = square of the sum of the first n numbers**:
  $1 + 8 + 27 + 64 = 100 = (1 + 2 + 3 + 4)^2$
- each L: side $k(k+1)/2$ square less side $(k-1)k/2$ square $= k^3$

Mathematical induction

Principle. $P(n)$ a statement about the natural number n. If

1. $P(1)$ is true (base step), and
2. for every k, $P(k)$ true $\Rightarrow$ $P(k+1)$ true (inductive step),

then $P(n)$ is true for every natural number n.

- dominoes: the first falls; each fall knocks the next
- checking many values proves nothing: $n^2 - n + 41$ is prime for $n = 1$ to 40, not at 41

Writing the proof (four steps):

1. $n = 1$: left side = right side
2. assume $P(k)$
3. add the $(k+1)$th term to both sides; factor **towards** $P(k+1)$'s right side (write it down first)
4. $P(1)$ true and $P(k) \Rightarrow P(k+1)$: true for all n

- step 3 for $\Sigma\ k^2$: $k(k+1)(2k+1)/6 + (k+1)^2 = (k+1)(2k^2+7k+6)/6 = (k+1)(k+2)(2k+3)/6$
- step 3 for $\Sigma\ k$: $k(k+1)/2 + (k+1) = (k+1)(k+2)/2$
- **to find** a sum: compute the first few, guess, prove the guess by induction
  - $1/(1\cdot 2) + 1/(2\cdot 3) + \cdots$: $1/2, 2/3, 3/4 \to n/(n+1)$
  - also telescopes: $1/k - 1/(k+1)$
- divisibility: $3^{k+1} - 1 = 3(3^k - 1) + 2$
- 4.2 **finds** a formula; induction **proves** a given one

COMMON MISTAKES

✗ $\Sigma\ k^2 = (\Sigma\ k)^2$
✓ Σ never splits over products; expand t_k first.

✗ $\Sigma\ (k^2 + 1) = \Sigma\ k^2 + 1$
✓ $\Sigma\ 1 = n$: $\Sigma\ k^2 + n$.

✗ inductive step only
✓ the base step too: a false formula can pass the step.

✗ $P(k)$ proved in step 2
✓ it is assumed; step 3 says where it is used.

✗ $11^2 + \cdots + 20^2$ = sum to 20 − sum to 11
✓ less the sum to **10**.

✗ a formula checked for $n = 1, 2, 3$ called proved
✓ prove it by induction.