Unit 3 · Complex Numbers

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Algebra (Units 1–5 together): 20 of 75 marks; 2 MCQs, 2 short and 1 long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's model paper:** a 2-mark part, "use De Moivre's theorem to solve $x^3 = -1$"
- **MCQ (1 mark)**
  - range of the principal argument
  - De Moivre's statement
  - polar form of $-3i$
  - quotient in polar form
  - a power by De Moivre
  - $(1 + \omega^2)^3$; products in ω
  - fourth roots of 16
  - modulus, argument of a quotient
  - $e^{i\pi/2} + e^{i\pi}$
- **Short and long (5 or 8 marks, in parts)**
  - polar form, then a power
  - solve $z^4 =$ a complex number
  - identities in ω
  - $z^n + 1/z^n = 2\cos n\theta$
  - exponential form; $\cos(\pi/12)$ from a product
  - $\cos 4\theta$ in powers of $\cos\theta$; roots as cosines

Key terms

Polar form: $z = r(\cos\theta + i\sin\theta)$, also written cis θ.

Modulus $r = |z| = \sqrt{a^2 + b^2}$: distance of (a, b) from O.

Argument $\arg z$: angle of OP from the positive real axis; many (add 2π); 0 has none.

Principal argument: the one with $-\pi < \theta \le \pi$.

Cube roots of unity: 1, ω, ω^2, the roots of $x^3 = 1$.

Exponential form: $z = re^{i\theta}$.

Formulas

$z = a + bi = r(\cos\theta + i\sin\theta)$	$a = r\cos\theta$, $b = r\sin\theta$ **Use:** polar form
$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$	moduli multiply, arguments add **Use:** product
$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$	divide, subtract **Use:** quotient
$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$	every integer n **Use:** De Moivre
$z^n = r^n(\cos n\theta + i\sin n\theta)$	raise r, multiply θ **Use:** powers
$e^{i\theta} = \cos\theta + i\sin\theta$	θ in radians **Use:** Euler's formula
$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$	add $e^{i\theta}$, $e^{-i\theta}$ **Use:** cosine
$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$	subtract **Use:** sine
$e^{i\pi} + 1 = 0$	$\theta = \pi$ **Use:** Euler's identity

nth roots ($k = 0, 1, \ldots, n-1$):

$$z^{1/n} = r^{1/n}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right]$$

Polar form: the argument

1. $r = \sqrt{a^2 + b^2}$
2. $\alpha = \tan^{-1}|b/a|$, the acute angle with the real axis
3. mark the quadrant of (a, b), then:

Quadrant	I	II	III	IV
$\arg z$	α	$\pi - \alpha$	$-(\pi - \alpha)$	$-\alpha$

- on the axes: $\pi/2$ ($a = 0, b > 0$); $-\pi/2$ ($b < 0$); π (negative real)
- $1 + i$: $\pi/4$; $-1 - i$: $-3\pi/4$ (same b/a, opposite ways)
- some books use 0 to 2π ($-3\pi/4 = 5\pi/4$): either, if the range is said
- $-1 + \sqrt{3}\,i = 2(\cos 2\pi/3 + i\sin 2\pi/3)$
- $-2 - 2i = 2\sqrt{2}[\cos(-3\pi/4) + i\sin(-3\pi/4)]$

Product, why: multiply out ($i^2 = -1$); the brackets are $\cos(\theta_1 + \theta_2)$, $\sin(\theta_1 + \theta_2)$ (compound angles)

- quotient: undoes the product

- multiplying by z: stretch by $|z|$, turn by $\arg z$; by i: a quarter turn

De Moivre's theorem

Proof for n a positive integer (induction):

1. $n = 1$: both sides $\cos\theta + i\sin\theta$
2. assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$
3. multiply by $\cos\theta + i\sin\theta$: product rule $\rightarrow$ $\cos(k+1)\theta + i\sin(k+1)\theta$
4. true for every positive integer

- $n = 0$: both sides 1
- $n = -m$: $1/(\cos m\theta + i\sin m\theta) = \cos m\theta - i\sin m\theta$ ($\times$ conjugate; bottom $\cos^2 + \sin^2 = 1$) = $\cos(-m\theta) + i\sin(-m\theta)$
- $n = p/q$: only **one** of the values (all of them: roots)
- $(1+\sqrt{3}\,i)^8 = 2^8(\cos 8\pi/3 + i\sin 8\pi/3) = -128 + 128\sqrt{3}\,i$
- **multiple angles**: expand $(c+is)^n$ by the binomial theorem; equate real and imaginary parts
  - $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
  - $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
  - $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$
- $z = \cos\theta + i\sin\theta$: $z^n + 1/z^n = 2\cos n\theta$

Roots of a complex number

1. want $w = \rho(\cos\varphi + i\sin\varphi)$ with $w^n = z$; De Moivre: $\rho^n(\cos n\varphi + i\sin n\varphi)$
2. equal: moduli equal, arguments differ by whole turns: $\rho^n = r$, $n\varphi = \theta + 2k\pi$
3. $\rho = r^{1/n}$ (positive real root), $\varphi = (\theta + 2k\pi)/n$
4. $k = 0$ to $n-1$: n different angles, $2\pi/n$ apart; $k = n$ repeats $k = 0$

- **exactly n roots** of any non-zero number
- corners of a **regular n-gon** on the circle of radius $r^{1/n}$, the first at θ/n
- to solve $z^n = c$: write c in polar form first
  - $z^4 = -16 = 16(\cos\pi + i\sin\pi) \rightarrow \sqrt{2}(\pm 1 \pm i)$: a square, two conjugate pairs
  - cube roots of $8i$: $\sqrt{3} + i$, $-\sqrt{3} + i$, $-2i$ (30°, 150°, 270°)

Cube roots of unity

$$1, \quad \omega = \frac{-1+\sqrt{3}\,i}{2}, \quad \omega^2 = \frac{-1-\sqrt{3}\,i}{2}$$

- from $\cos(2k\pi/3) + i\sin(2k\pi/3)$; or $x^3 - 1 = (x-1)(x^2+x+1)$
- corners of an equilateral triangle in the unit circle (0°, 120°, 240°)

1. $\omega^3 = 1$: powers repeat in threes; divide the power by 3, keep the remainder ($\omega^{3m+1} = \omega$)
2. $\mathbf{1 + \omega + \omega^2 = 0}$: three equal arrows a third of a turn apart cancel
3. each complex root is the square of the other: $(\omega^2)^2 = \omega$
4. $\omega \cdot \omega^2 = 1$: $\omega^2 = 1/\omega = \overline{\omega}$
5. cube roots of a real a: $\sqrt[3]{a}$, $\sqrt[3]{a}\,\omega$, $\sqrt[3]{a}\,\omega^2$
   - of -1: -1, $-\omega$, $-\omega^2$; of -8: -2, $1-\sqrt{3}\,i$, $1+\sqrt{3}\,i$
6. factors:
   - $a^3 - b^3 = (a-b)(a-b\omega)(a-b\omega^2)$
   - $a^3 + b^3 = (a+b)(a+b\omega)(a+b\omega^2)$
   - $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

Working rule: $1 + \omega = -\omega^2$; $1 + \omega^2 = -\omega$; $\omega + \omega^2 = -1$; powers by remainders

- $1 - \omega + \omega^2 = -2\omega$, $1 + \omega - \omega^2 = -2\omega^2$: product 4; fifth powers add to 32

Three phases: V, $V\omega^2$, $V\omega$ ($V = 230$ V)

- between two live wires: $|1-\omega| = \sqrt{3} \rightarrow 230\sqrt{3} \approx 398 \approx 400$ V
- neutral, balanced load: $I(1 + \omega + \omega^2) = 0$

Euler's formula

- put $x = i\theta$ in e^x's series; powers of i cycle $i, -1, -i, 1$
- real part = series for $\cos\theta$, imaginary = series for $\sin\theta$ (taken on trust)
- $re^{i\theta}$: products and powers become laws of indices
  - $r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$
  - $(e^{i\theta})^n = e^{in\theta}$ is De Moivre
- $1 - i = \sqrt{2}\,e^{-i\pi/4}$; $-2 = 2e^{i\pi}$; $e^{i\pi/3} \cdot e^{i\pi/6} = e^{i\pi/2} = i$
- compare $\cosh x = (e^x + e^{-x})/2$ (Unit 12)

COMMON MISTAKES

✗ $\arg(-1-i) = \tan^{-1}1 = \pi/4$
✓ quadrant III: $-3\pi/4$.

✗ the cube roots of $8i$: $2i$ only
✓ three roots, $k = 0, 1, 2$.

✗ $z^{1/n}$ modulus r
✓ $r^{1/n}$.

✗ $\omega^{10} = \omega^{10}$, left as it is
✓ remainder of $10 \div 3$: ω.

✗ $1 + \omega + \omega^2 = 1$
✓ 0.

✗ θ in degrees in $e^{i\theta}$
✓ in radians.

✗ k run to n
✓ $k = n$ repeats $k = 0$: stop at $n-1$.