Unit 2 · Binomial Theorem

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Algebra (Units 1–5 together): 20 of 75 marks; 2 MCQs, 2 short and 1 long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's model paper:** a short question of five 1-mark parts on $(a+b)^n$: the general term; the general term with $-a$ for a; the number of terms; an expansion (e^x); the relation between $C(n, r)$, $C(n, r-1)$, $C(n+1, r)$
- **MCQ (1 mark)**
  - number of terms
  - the series for e^x
  - $C_0 - C_1 + C_2 - \cdots$
  - a middle term, the 4th term
  - a coefficient in $(1-x)^{-3}$
  - when a fractional-index series is valid
  - $\sqrt{4.04}$, $(0.99)^{10}$ by approximation
  - sum of the odd-power coefficients
- **Short and long (5 or 8 marks, in parts)**
  - prove: even coefficients = odd = 2^{n-1}
  - $(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5$
  - term independent of x; coefficient of a power
  - Pascal's triangle row; n from three coefficients
  - expand $(1+x)^{1/3}$; a cube root to 4 places
  - define e; a^x; log 1.2 from the series
  - state and prove the binomial theorem; middle terms
  - show $\log_e 2 = 1 - 1/2 + 1/3 - \cdots$

Key terms

Binomial: a sum of two terms, $a + b$.

General term t_{r+1}: the term with b^r, the $(r+1)$th (counting starts at $r = 0$).

Binomial coefficients C_0, C_1, ..., C_n: the $C(n, r)$ in $(1+x)^n$.

Term independent of x: the term whose power of x is 0.

Euler's number e: the limit of $(1+1/n)^n$; $e = 1 + 1/1! + 1/2! + \cdots = 2.718281\ldots$

log: in this unit $\log_e$, as CDC writes it (Grade 11's ln).

Formulas

$(a+b)^n = \Sigma\, C(n, r)\, a^{n-r} b^r$	$r = 0$ to n; n positive integer **Use:** binomial theorem
$t_{r+1} = C(n, r)\, a^{n-r} b^r$	kth: $r = k-1$ **Use:** general term
$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \cdots$	n rational, $\lvert x\rvert < 1$ **Use:** any index
$t_{r+1} = \dfrac{n(n-1)\cdots(n-r+1)}{r!}x^r$	any index **Use:** general term
$(1+x)^n \approx 1 + nx$	x small **Use:** approximate

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	every x **Use:** e^x
$a^x = 1 + x\log a + \frac{(x\log a)^2}{2!} + \cdots$	$a > 0$ **Use:** a^x
$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$	$-1 < x \le 1$ **Use:** log series

By heart ($|x| < 1$):

$$(1-x)^{-1} = 1 + x + x^2 + \cdots \qquad (1+x)^{-1} = 1 - x + x^2 - \cdots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Binomial theorem: positive integer

$$(a+b)^n = C(n,0)a^n + C(n,1)a^{n-1}b + \cdots + C(n,r)a^{n-r}b^r + \cdots + C(n,n)b^n$$

Proof, by counting:

1. $(a+b)^n = n$ brackets multiplied; each product takes one letter from every bracket
2. b from r brackets, a from the other $n-r$: $a^{n-r}b^r$
3. which r brackets give b: $C(n,r)$ ways $\to a^{n-r}b^r$ comes $C(n,r)$ times
4. $r = 0, 1, \ldots, n$: all the terms ∎

(Induction also proves it, with Pascal's rule as the step.)

- **$n+1$ terms**; in each the powers add to n (a's falls, b's rises)
- coefficients read the same from both ends: $C(n,r) = C(n,n-r)$
- $(a-b)^n$: signs alternate $+ - + -$
- $(1+x)^n = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$
- **Pascal's triangle**: each number = sum of the two above ($C(n,r-1) + C(n,r) = C(n+1,r)$)
  - rows: 1 · 1 1 · 1 2 1 · 1 3 3 1 · 1 4 6 4 1 · 1 5 10 10 5 1 · 1 6 15 20 15 6 1
- check an expansion: put $x = 1$ on both sides ($(2x-3)^4$: both give 1)

General term

- **a given term**: the 6th has $r = 5$
- **a given power of** x: write the power of x in t_{r+1}, set it equal, solve for r
  - r not a whole number from 0 to $n \to$ **no such term**
- **independent of** x: power 0
  - $(x^2 - 2/x)^9$: power $18 - 3r$; $r = 6 \to$ 7th term $C(9,6)(-2)^6 = 5376$; no x^{10} term
- **middle term**: n even $\to$ one, $t_{n/2+1}$; n odd $\to$ two, $t_{(n+1)/2}$, $t_{(n+3)/2}$
- **rth from the end** $= (n-r+2)$th from the start (or expand $(b+a)^n$)
- keep the sign of b inside: $(-2/x)^r = (-2)^r x^{-r}$

Binomial coefficients

In $(1+x)^n = C_0 + C_1x + \cdots + C_nx^n$:

1. $C_r = C_{n-r}$; greatest in the middle
2. $x = 1$: $C_0 + C_1 + \cdots + C_n = 2^n$
3. $x = -1$: $C_0 - C_1 + C_2 - \cdots + (-1)^n C_n = 0$ $(n \ge 1)$
4. add 2 and 3: $C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + \cdots = 2^{n-1}$

- as many even-sized subsets as odd-sized
- **sum of the coefficients** of any expansion: every letter = 1
  - odd powers only: $[f(1) - f(-1)]/2$
- three coefficients in AP or given: use $C(n,r)/C(n,r-1) = (n-r+1)/r$
  - 5th, 6th, 7th of $(1+x)^n$ in AP $\to n = 7$ or 14 (both work)
- $(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5$: terms with r odd cancel, those with r even double

Any index

- n a positive integer: factor $n - n = 0$ from $r = n+1 \to$ series stops (2.1), true for every x
- otherwise **never stops**; equals $(1+x)^n$ **only for $|x| < 1$**
- $(a+b)^n = a^n(1+b/a)^n$, the larger term outside; valid for $|b/a| < 1$
- $(4+x)^{1/2} = 2(1+x/4)^{1/2}$, valid $|x| < 4$
- $(1-2x)^{-1/2} = 1 + x + 3x^2/2 + 5x^3/2 + \cdots$, valid $|x| < 1/2$
- Grade 11's infinite GP is $n = -1$: $(1-x)^{-1} = 1/(1-x)$

Approximation

- small x: x^2, x^3 much smaller; first 2 or 3 terms; error ≈ the first term left out

- roots: factor out the **nearest** perfect power, so the rest is $1 +$ (small)
  - $\sqrt{26} = 5(1 + 0.04)^{1/2} = 5(1 + 0.02 - 0.0002 + 0.000004) = 5.0990$
  - $\sqrt[3]{126} = 5(1 + 1/125)^{1/3}$
- larger $x \to$ more terms needed
  - $g' = g(1 + h/R)^{-2}$: Everest 2 terms; space station ($h/R = 0.0625$) 4 terms $\to 8.68$ m/s^2
- a small percentage error in $r \to$ about n times as large in r^n

Euler's number e

1. expand $(1 + 1/n)^n$ and divide each factor by an n:
$$1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots$$
2. **grows** with n: every bracket grows, and more terms are added
3. **stays below 3**: brackets < 1 and $r! \geq 2^{r-1} \to$ less than $1 + (1 + 1/2 + 1/4 + \cdots) = 3$
4. growing, bounded $\to$ a limit, e; $2 < e \leq 3$
5. each bracket $\to 1$: $e = 1 + 1/1! + 1/2! + 1/3! + \cdots$

- ten terms give $e = 2.718282$ (6 places); $(1 + 1/n)^n$ at $n = 10^6$ only 2.718280

e^x, a^x, log (1 + x)

- $e^x = \lim(1 + x/n)^n$; expand as for $e \to$ the series; true for **every** x ($r!$ outgrows x^r)
- $x = -1$: $e^{-1} = 1 - 1 + 1/2! - 1/3! + \cdots$
- $a^x = e^{x \log a}$: the same series with $x \log a$
- **log series**: $(1 + x)^y$ two ways (binomial; $e^{y \log(1+x)}$); equate the coefficients of y
- $x = 1$: $\log_e 2 = 1 - 1/2 + 1/3 - 1/4 + \cdots$ (true but slow: 1000 terms give 0.6926)
- small x is quick: $\log 1.1 = 0.1 - 0.005 + 0.000333 - \cdots = 0.0953$
- x negative: every term of $\log(1 + x)$ negative ($\log 0.9 = -0.1054$)
- **rule of 72**: doubling at R% needs $n = \log 2/\log(1 + r) \approx 69.31/R + 0.35 \approx 72/R$ near 8%

COMMON MISTAKES

✗ $(4 + x)^{1/2} = 4(1 + x/4)^{1/2}$
✓ $4^{1/2} = 2$ outside: $2(1 + x/4)^{1/2}$.

✗ $(4 + x)^{1/2}$ valid for $|x| < 1$
✓ the bracket's $x/4$ must be < 1 in size: $|x| < 4$.

✗ the 6th term has $r = 6$
✓ t_{r+1}: $r = 5$.

✗ $(2x - 1)^6$'s terms all positive
✓ $b = -1$: powers of (-1) alternate the signs.

✗ a fractional-index series used at any x
✓ only for $|x| < 1$.

✗ $(1 - 3x)^{1/2}$ valid for $|x| < 1$
✓ $|3x| < 1$: $|x| < 1/3$.

✗ $C_0 - C_1 + \cdots = 2^n$
✓ that is $x = 1$; $x = -1$ gives 0.