Unit 1 · Permutation and Combination

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Algebra (Units 1–5 together): 20 of 75 marks; 2 MCQs, 2 short and 1 long question
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours
- **CDC's model paper:** MCQ: arrangements of n things r at a time, repetition allowed (n^r); short: the relation between $C(n, r)$, $C(n, r-1)$ and $C(n+1, r)$; long: the letters of INTERVAL (vowels together; vowels and consonants keep their places; vowels only in odd places)
- **MCQ (1 mark)**
  - a formula: $P(n, r)$, $C(n, r)$, round table
  - letters of a word with repeats
  - n from $C(n, 2)$
  - diagonals of a polygon
  - non-empty selections
  - circle or necklace
  - arrange or choose?
- **Short and long (5 or 8 marks, in parts)**
  - prove Pascal's rule; $r\,C(n, r) = n\,C(n-1, r-1)$
  - $C(n, x) = C(n, y)$: find n
  - letters: together, never together, fixed ends, own places
  - a committee or team with "at least"
  - round table: no two men together
  - digits with repetition: at least one repeated, even
  - the rank of a word in dictionary order

Key terms

Basic principle of counting: one thing in m ways and **then** another in n ways (whatever came first): $m \times n$ ways.

Addition principle: m ways **or** n other ways, none counted in both: $m + n$.

Permutation: an **arrangement**; order matters ($12 \neq 21$).

Combination: a **selection**; order doesn't matter (the XI, not the batting order).

Factorial $n!$: $n(n-1)\cdots 2 \cdot 1$; $n! = n \cdot (n-1)!$; $0! = 1$ (one way to arrange nothing).

Formulas

$P(n, r) = \dfrac{n!}{(n-r)!}$	n different, r at a time, $r \leq n$ **Use:** arrange; $P(n, n) = n!$
$\dfrac{n!}{p!\,q!\,r!}$	p, q, r alike **Use:** not all different
$(n-1)!$	n different **Use:** in a circle (table)
$\dfrac{1}{2}(n-1)!$	$n \geq 3$, can be turned over **Use:** necklace, garland
n^r	each place any of n; r may be $> n$ **Use:** repetition allowed
$C(n, r) = \dfrac{n!}{r!\,(n-r)!}$	$= P(n, r)/r!$ **Use:** choose
$\dfrac{n(n-3)}{2}$	$C(n, 2) - n$ **Use:** diagonals of an n-gon

- also written ${}^nP_r$, ${}^nC_r$ and n over r in brackets: all mean the same
- $P(n, r) = n(n-1)\cdots(n-r+1)$: r factors, one fewer each place

- $P(n, r) = n \cdot P(n-1, r-1)$: fill the first place, arrange the rest

Counting: "and then", "or"

- "and then", same number of choices whatever came first: **multiply**
- "or", no overlap: **add**; overlap: take it away ($n(A \cup B) = n(A) + n(B) - n(A \cap B)$)
- **fill the restricted place first**
- restrictions touching each other (first not 0, last even): split into cases
  - four-digit even numbers from 0–6, no repeats: last 0 $\rightarrow 6 \times 5 \times 4 = 120$; last 2, 4, 6 $\rightarrow 3 \times 5 \times 5 \times 4 = 300$; total **420**
- **"at least one"** = all − "none": PINs with a repeated digit $10^4 - P(10, 4) = 4960$
- which thing chooses, from how many? (letters into boxes: each letter picks a box)

Arrangements with conditions

- **together**: tie them into one block; arrange the blocks; then arrange inside the block
- **never together** (two objects): all − together
- **no two together** (several): place the others first, then put them in the gaps (k in a row leave $k+1$ gaps; round a table, k)
- **fixed place**: put that object there first, then the rest
- **vowels in the odd places**: fill those places first, P(places, vowels)
- **own places** (vowels where vowels were): arrange vowels among themselves × consonants among themselves

NEPAL: $5! = 120$; EA together $4! \times 2! = 48$; not together 72; vowels in odd places $P(3, 2) \times 3! = 36$

BHAKTAPUR (A twice): $9!/2! = 1{,}81{,}440$

- vowels together: $7! \times 3!/2! = 15{,}120$
- no two vowels together: $6! \times P(7, 3)/2! = 720 \times 105 = 75{,}600$

Rank in dictionary order (POKHARA is 2084th): letters in order A, A, H, K, O, P, R; count the words starting with each smaller letter, then with P and a smaller second letter, and so on; the word itself + 1

Shortest routes a east, b north: $(a+b)!/(a!\,b!)$ (4 E's, 3 N's: 35)

Not all different: why $n!/(p!\,q!\,r!)$

1. label the p alike ones 1, …, p: $p!$ relabelings, each a new arrangement
2. same for q, r: each arrangement gives $p!\,q!\,r!$ of n different
3. $x \cdot p!\,q!\,r! = n!$

Circular arrangement

- cut a circle at any of its n places $\rightarrow n$ rows: $n \cdot x = n! \rightarrow x = (n-1)!$
- or: fix one person's seat, arrange the other $n-1$ in a row
- necklace, garland: turned over, clockwise = anticlockwise $\rightarrow$ **half**
- people at a table can't be turned over: no halving
- 8 at a table: $7! = 5040$; two together: $6! \times 2 = 1440$
- 6 men, 6 women, no two men together: women 5!, men in the 6 seats between $6! \rightarrow 86{,}400$

Combination

1. an arrangement = **select** r ($C(n, r)$) **and then** order them ($r!$)
2. $P(n, r) = C(n, r) \times r! \rightarrow C(n, r) = n!/[r!\,(n-r)!]$

- swap two chosen: a different result? yes $\rightarrow P$ (captain, vice-captain); no $\rightarrow C$ (committee)
- choose the people, then arrange them: $C \times$ arrangements
- **"at least 2 girls"**: add the cases (2, 3, 4 girls), each girls × boys; or total − (0 or 1 girl)
  - 5 from 6 boys, 4 girls: $120 + 60 + 6 = 186 = 252 - 66$
- round-robin league: C(teams, 2) matches (8 teams: 28)
- triangles from n points, k on a line: $C(n, 3) - C(k, 3)$ (12 points, 5 on a line: 210)
- rectangles on a chessboard: 2 of the 9 lines each way, $C(9, 2)^2 = 1296$
- squad: $C(15, 11) = 1365$ elevens; batting orders $P(15, 11) = 1365 \times 11!$
- $C(n, r)$ counts something $\rightarrow$ whole number $\rightarrow$ product of r consecutive whole numbers divisible by $r!$

Properties of combination

Prove by **meaning** (count one thing two ways) or by the **formula**; either is accepted.

1. $C(n, 0) = C(n, n) = 1$
2. $C(n, r) = C(n, n - r)$: choosing who goes = choosing who stays
   - $C(n, x) = C(n, y) \to x = y$ or $x + y = n$ ($C(n, 8) = C(n, 6) \to n = 14$)
3. **Pascal's rule**: $C(n, r) + C(n, r - 1) = C(n + 1, r)$, $1 \le r \le n$
   - meaning: Pasang out $\to C(n, r)$; Pasang in $\to C(n, r - 1)$
   - formula: common denominator $r!\,(n - r + 1)!$; top $n!\,[(n - r + 1) + r] = (n + 1)!$
   - builds Pascal's triangle (Unit 2)
4. $r\,C(n, r) = n\,C(n - 1, r - 1)$: committee, then its chair = chair first, then the rest
   - hence the sum of $r\,C(n, r)$ for $r = 1$ to n is $n \cdot 2^{n-1}$
5. $C(n, r)/C(n, r - 1) = (n - r + 1)/r$: > 1 while $r < (n + 1)/2$
   - greatest at $r = n/2$ (n even), at $(n - 1)/2$ and $(n + 1)/2$ (n odd)
6. $C(n, 0) + C(n, 1) + \cdots + C(n, n) = 2^n$: all subsets (each element in or out)
   - at least one of n things: $2^n - 1$

COMMON MISTAKES

✗ 4 letters into 3 boxes: 4^3
✓ each letter chooses a box: $3^4 = 81$.

✗ MAHENDRANAGAR: 13!/(4! + 2! + 2!)
✓ divide by the **product** 4! 2! 2!.

✗ 7 people at a table: ½ × 6!
✓ 6!; halve only a necklace or garland.

✗ no two vowels together = all − all vowels together
✓ place the consonants, vowels into the gaps.

✗ $C(n, 2) = 45$: $n = 10$ or -9
✓ only 10; a count is never negative.

✗ non-empty selections of 6 fruits: $2^6 = 64$
✓ $2^6 - 1 = 63$.

✗ diagonals of a hexagon: $C(6, 2) = 15$
✓ less the 6 sides: 9.