Unit 21

Potential, Potential Difference and Potential Energy

This unit online, with its interactives:
science-playground.com/s/11f21

A bird sits on a bare power line carrying 11,000 volts and preens its feathers, quite unharmed. You walk across a carpet on a dry winter day, reach for a door handle, and a spark a few thousand volts strong snaps at your finger. Twelve volts from a car battery can't be felt at all, yet the same twelve volts can start an engine. What is a volt, then, and why does the number alone tell you so little about danger?

The answer is **electric potential**: the energy a charge has per coulomb because of where it is in an electric field. In Unit 19 you met electric charges and Coulomb's law, and in Unit 20 the electric field, the force per unit charge. This unit does for electricity what the gravitational potential did for gravity in Unit 7: it replaces forces, which are vectors and hard to add, with energies per unit charge, which are plain numbers. You will find the potential of a point charge, see how the field and the potential are two descriptions of the same thing, draw maps of equal potential, and use the **electron volt**, the energy unit of atoms. The capacitors of Unit 22 rest on all of it.

21.1 Electrical potential

The **electric potential** V at a point is the **work done in bringing a unit positive test charge from infinity to that point** (without accelerating it, so that all the work is stored). If bringing a charge Q there takes work W,

$$V = \frac{\text{work done in bringing the charge from infinity}}{\text{charge}} = \frac{W}{Q}$$

Its SI unit is the joule per coulomb (J/C), called the **volt** (V): a point is at 1 V when it takes 1 J of work to bring 1 C there from infinity. Potential is a **scalar**: it has a size and a sign, but no direction.

Because the work done is stored as potential energy, there is a second way to say the same thing: **the potential at a point is the electric potential energy per unit charge that a charge placed there would have**. A charge Q at a point of potential V has potential energy QV.

The definition follows the gravitational potential exactly (Unit 7's 7.6 Gravitational potential, the work per kilogram to bring a mass from infinity), and "infinity" means the same thing: a place so far away that the force is zero there, where we take the potential to be zero. The difference is that gravity only attracts, while charges can attract or repel, so the electric potential can be negative **or positive**.

Opposite charges attract, and the situation is exactly as in gravitation. A negative charge near a positive one is held by the attraction: to pull it away to infinity you must do work. So, near a positive charge, a negative charge has less energy than it would have far away: its potential energy is **negative**, and it climbs to zero as the charges are separated.

Like charges repel. Imagine two negative charges held some distance apart. Bringing them together from infinity took work, against the push, and that work was stored in the pair. Let them go and they fly apart: the stored energy turns into their kinetic energy, until they are so far apart that the push is zero.

Their potential energy was **positive**: it rose above zero as they were pushed together, and falls back to zero as they fly apart.

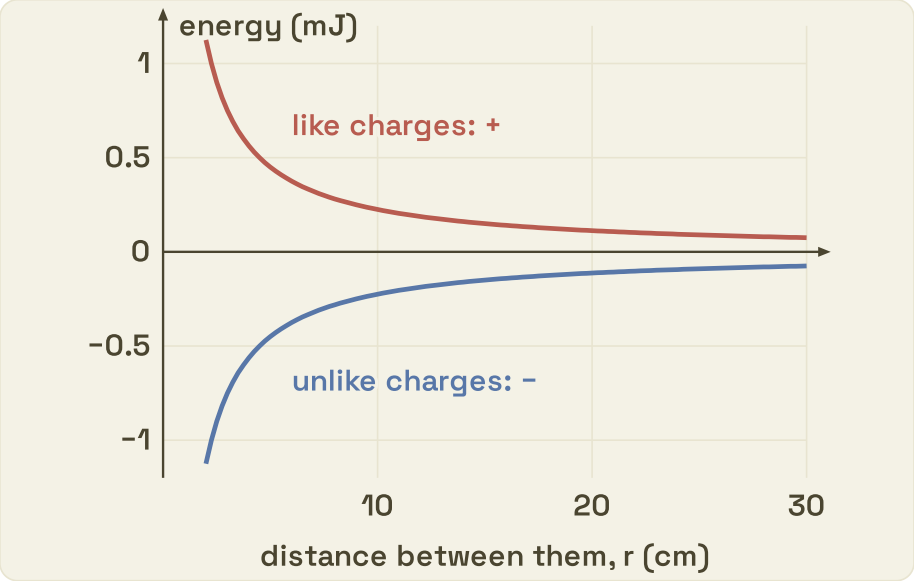


Figure 21.1 *The energy stored by two charges of 50 nC against the distance between them. For like charges (positive energy) it falls as they fly apart; for unlike charges (negative energy) it rises towards zero as they are pulled apart. Either way, zero is "far apart".*

CAREFUL

Potential is not potential energy

The **potential** V belongs to a point in a field: volts, joules per coulomb, whatever charge (if any) is put there. The **potential energy** QV belongs to a charge at that point: joules. Two coulombs at a point of 6 V have 12 J; one coulomb at the same point has 6 J. The point's potential is 6 V either way.

Check 1

A charge of +2.0 μC is brought from infinity to a point P. The work done on it is 8.0×10^{-4} J. The potential at P is:

a 1.6×10^{-9} V
b 4.0×10^{-10} V
c 2.5×10^{-3} V
d 400 V

Check 14

18. A proton and an electron are in a constant electric field created by oppositely charged plates. You release the proton from near the positive plate and the electron from near the negative plate. Which feels the larger electric force?

a The proton.
b The electron.
c Neither, there is no force.
d The magnitude of the force is the same for both and in the same direction.
e The magnitude of the force is the same for both but in opposite directions.

23. In the Bohr model of the hydrogen atom, an electron orbits a proton (the nucleus) in a circular orbit of radius 0.53×10^{-10} m. (a) What is the electric potential at the electron's orbit due to the proton? (b) What is the kinetic energy of the electron? (c) What is the total energy of the electron in its orbit? (d) What is the ionization energy, that is, the energy required to remove the electron from the atom and take it to infinity, at rest?

24. How much voltage must be used to accelerate a proton (radius 1.2×10^{-15} m) so that it has sufficient energy to just "touch" a silicon nucleus? A silicon nucleus has a charge of $+14e$ and its radius is about 3.6×10^{-15} m. Assume the potential is that for point charges.

25. An electron is liberated from a hot filament, and attracted by an anode, of potential 1200 volts positive with respect to the filament. What is the speed of the electron when it strikes the anode? ($m_e = 9.1 \times 10^{-31}$ kg)

26. A 9.0 V battery is connected across two large parallel plates that are separated by 4.5 mm of air, creating a potential difference of 9.0 V between the plates. (a) What is the electric field in the region between the plates? (b) An electron is released from rest at the negative plate. If the only force on the electron is the electric force exerted by the electric field of the plates, what is the speed of the electron as it reaches the positive plate? The mass of an electron is $m_e = 9.11 \times 10^{-31}$ kg.

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

A STORY FROM SCIENCE

Volta's pile, and the volt

In 1800 Alessandro Volta, a professor at Pavia in Italy, wrote to the Royal Society in London about a pile of zinc and copper discs, each pair separated by card soaked in salt water. Touch both ends and you felt a shock; join them with a wire and a steady stream of electricity flowed, for the first time not a single spark from a rubbed rod but a current that kept going. It was the first battery. In 1881 an international congress of electricians in Paris named the unit of potential difference after him: the volt.

21.2 The potential due to a point charge

Let us find the potential at a distance r from a single point charge Q, in free space (air is nearly the same).

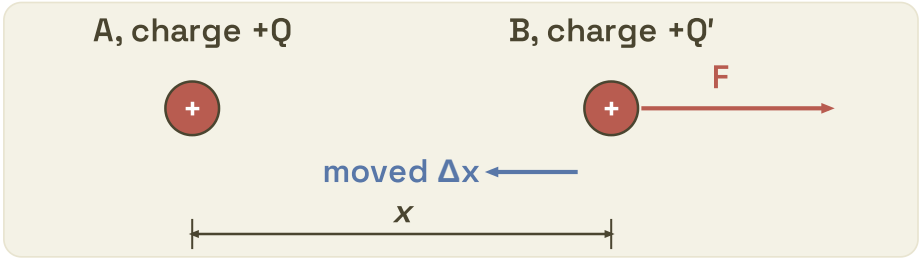


Figure 21.2 *A point charge $+Q$ at A and a second charge $+Q'$ at B, a distance x away. A pushes B away with a force F. An outside agent moves B a small step Δx towards A, against F.*

Derivation: $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r}$

Measure distances x from A, where the charge Q is (Figure 21.2).

1. By Coulomb's law, the charge Q at A pushes the charge Q' at B away with a force

$$F = \frac{1}{4\pi\varepsilon_0}\frac{QQ'}{x^2}$$

2. An outside agent moves the charge at B a small distance Δx towards A. The step is so small that F hardly changes over it. The agent pushes against F, so it does work. As B moves towards A, x gets smaller: Δx is a negative number, and the work done by the agent is

$$\Delta W = -F\,\Delta x = -\frac{1}{4\pi\varepsilon_0}\frac{QQ'}{x^2}\,\Delta x$$

The minus sign makes ΔW positive, because the motion is opposite to F.

3. Add up the work from infinity ($x = \infty$) to a distance r from A ($x = r$). In the limit of very small steps the sum is an integral:

$$W = -\frac{QQ'}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{dx}{x^2} = -\frac{QQ'}{4\pi\varepsilon_0}\left[-\frac{1}{x}\right]_{\infty}^{r} = -\frac{QQ'}{4\pi\varepsilon_0}\left(-\frac{1}{r}+0\right) = \frac{QQ'}{4\pi\varepsilon_0\,r}$$

4. If Q' is a unit positive charge, this work is, by definition, the potential at that distance:

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{kQ}{r}} \qquad k = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9 \text{ N m}^2/\text{C}^2$$

Compare Unit 7's derivation of $V = -GM/r$: the same steps, the same integral. Gravity's minus sign is there because gravity attracts; here, with Q positive, the force repels and the work done is positive.

A charge of 1.0 nC gives, at 10 cm, a potential of $V = \dfrac{8.99 \times 10^9 \times 1.0 \times 10^{-9}}{0.10} = 90$ V.

Four things the formula tells you

1. **Potential is a scalar**, so the potential at a point due to several point charges is the **algebraic sum** of the potentials due to each, signs included. There are no angles and no components to work out: this is why potentials are so much easier to handle than fields.
2. **The sign follows the charge**: the potential due to a positive charge is positive, and that due to a negative charge is negative.
3. **All points at the same distance from a point charge are at the same potential**. They lie on a sphere round the charge. A surface over which the potential is the same everywhere is called an **equipotential surface**, and at every point of one the electric field is **perpendicular** to the surface (21.5).
4. **The potential energy** of a charge Q' at a distance r from a point charge Q is the work done to bring it there:

$$E_p = \frac{1}{4\pi\varepsilon_0}\frac{QQ'}{r}$$

positive for like charges, negative for unlike ones, as in 21.1.

WORKED EXAMPLE

Two equal charges

Two charges of +2.0 nC are 20 cm apart. Find a. the potential at the point M halfway between them, b. the work needed to bring a +1.0 nC charge from infinity to M, and c. the electric field at M.

Answer

a. Each charge is 0.10 m from M. Potentials add as numbers:

$$V_M = \frac{kQ}{r} + \frac{kQ}{r} = 2 \times \frac{8.99 \times 10^9 \times 2.0 \times 10^{-9}}{0.10} = 2 \times 180 = \mathbf{360\ V}$$

b. $W = qV_M = 1.0 \times 10^{-9} \times 360 = \mathbf{3.6 \times 10^{-7}\ J}$.

c. Each charge pushes a test charge at M away from itself with a field of the same size, kQ/r^2, in opposite directions: the field at M is **zero**.

So the potential at M is not zero although the field is. Potential and field are different things: the field is about how the potential **changes** from point to point (21.3).

Check 2

A charge of +4 nC and a charge of −4 nC are 30 cm apart. At the point halfway between them:

a the potential and the field are both zero
b the potential is zero but the field is not
c the field is zero but the potential is not
d neither is zero

10. An electron is accelerated from rest by a potential difference of 0.10 V. How much greater would its final speed be if it is accelerated with four times as much voltage? Explain.
11. Is there a point along the line joining two equal positive charges where the electric field is zero? Where the electric potential is zero? Explain.
12. Can a particle ever move from a region of low electric potential to one of high potential and yet have its electric potential energy decrease? Explain.
13. If V = 0 at a point in space, must E = 0 there? If E = 0 at some point, must V = 0 at that point? Explain. Give examples for each.
14. Is the electric potential energy of two isolated unlike charges positive or negative? What about two like charges? What is the significance of the sign of the potential energy in each case?

Choose the correct option, with a reason

Check 11

15. A +0.2 μC charge is in an electric field. What happens if that charge is replaced by a +0.4 μC charge?

a The electric potential doubles, but the electric potential energy stays the same.
b The electric potential stays the same, but the electric potential energy doubles.
c Both the electric potential and electric potential energy double.
d Both the electric potential and electric potential energy stay the same.

Check 12

16. Which of the following statements is valid?

a If the potential at a particular point is zero, the field at that point must be zero.
b If the field at a particular point is zero, the potential at that point must be zero.
c If the field throughout a particular region is constant, the potential throughout that region must be zero.
d If the potential throughout a particular region is constant, the field throughout that region must be zero.

Check 13

17. Two identical positive charges are placed near each other. At the point halfway between the two charges,

a the electric field is zero and the potential is positive.
b the electric field is zero and the potential is zero.
c the electric field is not zero and the potential is positive.
d the electric field is not zero and the potential is zero.
e None of these statements is true.

Numerical problems

19. How much work does the electric field do in moving a −7.7 μC charge from ground to a point whose potential is 65 V higher?
20. How strong is the electric field between two parallel plates 6.8 mm apart if the potential difference between them is 220 V?
21. What potential difference is needed to stop an electron that has an initial velocity of 6.0×10^5 m/s? ($m_e = 9.11 \times 10^{-31}$ kg)

Check 9

1. The electric field between two charged plates is known to be 650 N/C. If the distance between the plates is 12 cm, what is the potential difference between the plates?

a 54 V
b 78 V
c 4500 V
d 7800 V

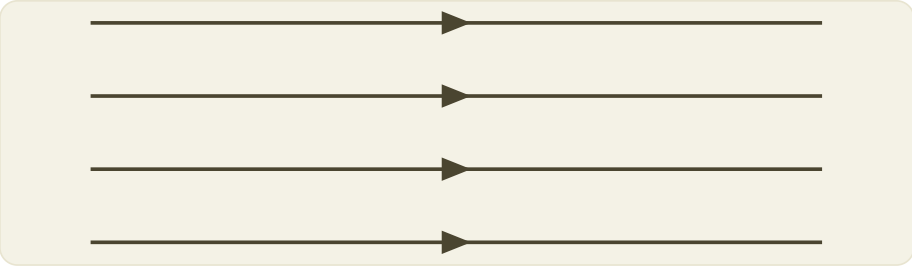


Figure 21.17 *For question 2: four straight, parallel, equally spaced field lines, all pointing the same way.*

Check 10

2. The field in the figure:

a could only be a gravitational field
b could only be a magnetic field
c could only be an electric field
d could be any one of gravitational, magnetic or electric fields

7. If two points are at the same potential, does this mean that no net work is done in moving a test charge from one point to the other? Does this imply that no force is exerted? Explain.
8. If a negative charge is initially at rest in an electric field, will it move toward a region of higher potential or lower potential? What about a positive charge? How does the potential energy of the charge change in each instance? Explain.
9. State clearly the difference (a) between electric potential and electric field, (b) between electric potential and electric potential energy.

21.3 Relation between E and V

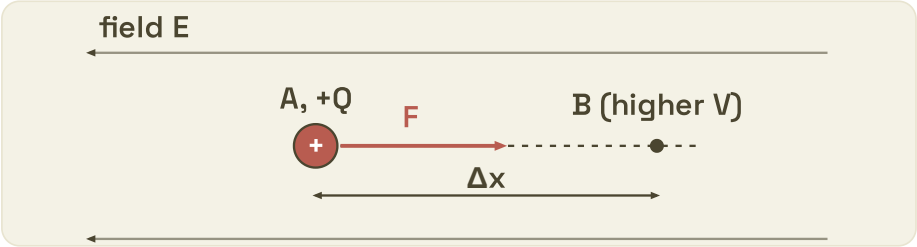


Figure 21.3 *A small charge +Q at A in an electric field E, which points to the left. An outside force F moves it to the right, against the field, a short distance Δx to B, which is at a higher potential.*

Suppose a force F moves a charge $+Q$ from a point A in an electric field E to a nearby point B, a short distance Δx along the x-axis (Figure 21.3). The distance is so small that the field, and the force, are the same all the way from A to B.

1. The work done by the force is
$$\Delta W = F\,\Delta x$$
2. The field pushes the charge with a force EQ. To move the charge against the field without accelerating it, the outside force must be equal and **opposite** to that push:
$$F = -EQ$$
(the minus sign: we are working out the work done by an outside agent against the field). So
$$\Delta W = -EQ\,\Delta x$$
3. If the potential difference between A and B is ΔV (B being at the higher potential), the work done in carrying Q from A to B is, by the definition of potential,
$$\Delta W = Q\,\Delta V$$
4. The two are the same work: $-EQ\,\Delta x = Q\,\Delta V$, so
$$\boxed{E = -\frac{\Delta V}{\Delta x}}$$

The ratio $\dfrac{\Delta V}{\Delta x}$ is the rate at which the potential changes with distance at a point in the field: it is called the **potential gradient** there. So **the electric field strength is the negative of the potential gradient**. The minus sign says that the field points the way the potential **falls**: if V increases in the positive direction of x, as it does going from A to B, the field points in the negative direction of x, from B to A. A positive charge, left alone, moves "downhill" in potential, the way a ball rolls downhill.

This also explains why the field strength has two units. From $E = -\Delta V/\Delta x$, its unit is the volt per meter, V/m. And
$$1\,\frac{\text{V}}{\text{m}} = 1\,\frac{\text{J/C}}{\text{m}} = 1\,\frac{\text{N m}}{\text{C m}} = 1\,\frac{\text{N}}{\text{C}}$$
so V/m and N/C are the same unit, as Unit 20 claimed.

The potential along that line, worked out point by point, shows both zeros, and the field as the slope:

Check 3

Along a line, the potential rises from 20 V to 50 V over 0.15 m, at a steady rate. The electric field along the line is:

a 200 V/m, pointing towards the higher potential
b 200 V/m, pointing towards the lower potential
c 4.5 V/m, pointing towards the lower potential
d 333 V/m, pointing towards the higher potential

WORKED EXAMPLE

Worked example: where is the potential zero?

Two charges, 3×10^{-8} C and -2×10^{-8} C, are 15 cm apart. At what point on the line joining the two charges is the electric potential zero?

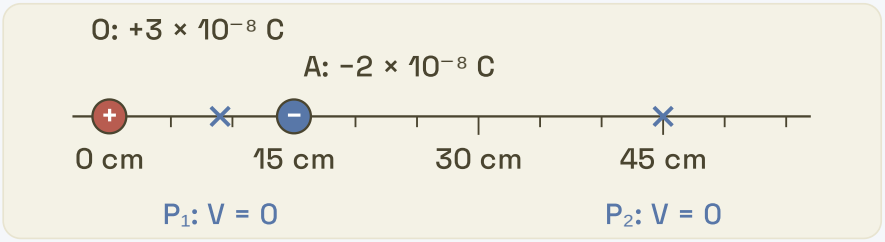


Figure 21.4 *The charges $+3 \times 10^{-8}$ C at O and -2×10^{-8} C at A, 15 cm apart. The potential is zero at P_1, 9 cm from O, and at P_2, 45 cm from O.*

Answer

Let P, the point where the potential is zero, be a distance x cm from the positive charge at O.

i. P between O and A. P is x cm from O and $(15 - x)$ cm from A. The total potential is zero:

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{3 \times 10^{-8}}{x \times 10^{-2}} - \frac{2 \times 10^{-8}}{(15 - x) \times 10^{-2}}\right] = 0$$

Everything outside the bracket, and the powers of ten inside it, cancel:

$$\frac{3}{x} - \frac{2}{15 - x} = 0 \;\Rightarrow\; 3(15 - x) = 2x \;\Rightarrow\; 45 = 5x \;\Rightarrow\; x = \mathbf{9\ cm}$$

ii. P on the line OA extended, beyond A. Now P is $(x - 15)$ cm from A:

$$\frac{3}{x} - \frac{2}{x - 15} = 0 \;\Rightarrow\; 3(x - 15) = 2x \;\Rightarrow\; x = \mathbf{45\ cm}$$

(On the other side of O, both charges are on the same side of P and the bigger, positive one is nearer, so the potential there is always positive.)

Thus the electric potential is zero at **9 cm** and **45 cm** from the positive charge.

21.4 Potential difference and the electron volt

In a circuit, and in most experiments, what matters is not the potential at one point but the **difference** between two. The **potential difference** between two points A and B is the work done in carrying a unit positive charge from A to B:

$$\Delta V = V_B - V_A = \frac{W_{A \to B}}{Q}$$

measured in volts. A **potential difference of 1 V** between two points means that 1 J of work is done in carrying 1 C from one to the other. The battery that is "9 V" keeps its + terminal 9 V above its − termi-

The 10^{36} comes from comparing the two forces between two protons, $\frac{ke^2}{Gm_p^2} = \frac{8.99 \times 10^9 \times (1.60 \times 10^{-19})^2}{6.67 \times 10^{-11} \times (1.67 \times 10^{-27})^2} \approx 1.2 \times 10^{36}$, the same at any distance, since both forces fall as $1/r^2$. Why, then, does gravity rule the Moon and the planets? Because big bodies have almost exactly as much positive as negative charge, and their electric forces cancel, while every kilogram adds to gravity. Why Is Gravity So Weak? tells that story.

Check 8

Which of these is true of gravitational fields but not of electric fields?

a they obey an inverse-square law
b they have infinite range
c the potential near an isolated body is always negative
d the field is minus the potential gradient

IN SHORT

Key ideas of Unit 21

- **Electric potential** at a point: the work done per unit positive charge in bringing a charge from infinity, $V = W/Q$, in volts (1 V = 1 J/C); a scalar, zero at infinity. It is also the potential energy per unit charge there.
- Point charge: $V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{kQ}{r}$, positive for a positive charge, negative for a negative one; potentials of several charges add as numbers.
- Potential energy of two charges: $E_p = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r} = qV$: positive for like charges, negative for unlike.
- Field and potential: $E = -\frac{\Delta V}{\Delta x}$, minus the **potential gradient**; the field points the way the potential falls; V/m = N/C. Uniform field between plates: $E = V/d$.
- **Potential difference** $\Delta V = W/Q$; the work done by the field is $-q\,\Delta V$. **1 eV** = 1.60×10^{-19} J, the energy a charge e gains through 1 V. Electron gun: $\frac{1}{2}mv^2 = eV$, $v = \sqrt{2eV/m}$.
- **Equipotentials** are perpendicular to field lines; no work is done along one; close equipotentials mean a strong field.
- The area under an E–r graph is the change in potential; the slope of a V–r graph is −E.
- Gravitational and electric fields: both inverse-square, infinite range; gravity attracts only and is about 10^{36} times weaker between two protons.

Exercise

Choose the correct option

Answer these questions

3. Explain why the test charges we use when measuring electric fields must be small.
4. When determining an electric field, must we use a positive test charge, or would a negative one do as well? Explain.
5. Why can electric field lines never cross?

Check 7

The potential at a distance r from a point charge is 120 V and the field there is 600 V/m. The distance r is:

a 5.0 m
b 0.20 m
c 72,000 m
d 2.0 m

21.7 Similarities between electric and gravitational fields

Newton's law of gravitation and Coulomb's law have the same shape, so the two forces behave alike in many ways. There are also important differences.

The main **similarities**:

1. Both are **non-contact** forces: they act across a distance, without anything touching.
2. Both have **infinite range**.
3. Both obey **inverse-square laws**.

The main **differences**:

1. Gravitational forces between masses are **always attractive**; electric forces can be attractive or repulsive.
2. An electric force is **much stronger** than the gravitational force: between two protons, about 10^{36} times stronger.
3. An electric force can be **shielded** (a metal box keeps it out); nothing shields gravity.
4. An electric force acts only on **charged** objects; the gravitational force acts on **all** objects.

Characteristic	Gravitation	Electric
Acts on	mass (positive only)	charge (positive or negative)
Force (N)	$F = \dfrac{GMm}{r^2}$, attractive only, infinite range	$F = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Qq}{r^2}$, attractive or repulsive, infinite range
Relative strength	1	10^{36}
Field strength	$g = \dfrac{F}{m}$	$E = \dfrac{F}{Q}$
Units	N/kg	N/C or V/m
Radial field	$g = \dfrac{GM}{r^2}$	$E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2} = \dfrac{kQ}{r^2}$
Potential difference	$\Delta V = \dfrac{\Delta W}{m}$	$\Delta V = \dfrac{\Delta W}{Q}$
Units	J/kg	J/C (volt)
Potential gradient	$g = -\dfrac{\Delta V}{\Delta r}$	$E = -\dfrac{\Delta V}{\Delta r}$
Potential in radial fields	$V = -\dfrac{GM}{r}$, $V = 0$ at ∞	$V = \dfrac{Q}{4\pi\varepsilon_0 r}$, $V = 0$ at ∞
Potential energy	$E_p = -\dfrac{GMm}{r}$	$E_p = \dfrac{Qq}{4\pi\varepsilon_0 r}$

nal: every coulomb that goes round a circuit from + to − can give 9 J to the circuit.

When a charge q goes from A to B:

- the work done by an outside agent (if it moves without speeding up) is $W = q\,\Delta V$;
- its potential energy changes by $\Delta E_p = q\,\Delta V$;
- the work done **by the field** is the opposite, $-q\,\Delta V$. If the charge is free, that work becomes kinetic energy: $\Delta E_k = -q\,\Delta V$.

Signs matter. A positive charge going to a higher potential ($\Delta V > 0$) gains potential energy, so it must be pushed. An electron, $q = -e$, going to a higher potential **loses** potential energy, so the field speeds it up: electrons fall "uphill" in potential.

The electron volt

Atoms, electrons and nuclei deal in energies so small that the joule is clumsy. Physicists use the energy an electron gains in falling through 1 V:

$$\textbf{1 electron volt} = 1\text{ eV} = e \times 1\text{ V} = 1.60 \times 10^{-19}\text{ J}$$

The **electron volt** is a unit of **energy**, not of potential: it is the kinetic energy a particle with charge e (an electron or a proton) gains when it is accelerated through a potential difference of 1 V. A charge e accelerated through V volts gains V eV. Its multiples are the keV (10^3 eV), MeV (10^6 eV), GeV (10^9 eV) and TeV (10^{12} eV).

Something	About
a photon of visible light	2 to 3 eV
to pull the electron out of a hydrogen atom	13.6 eV
an electron in an old television's picture tube	25 keV
the electrons in a dental X-ray machine	60 to 70 keV
the energy given out when one uranium nucleus splits	200 MeV
a proton in the Large Hadron Collider, near Geneva (since 2022)	6.8 TeV

The electron gun

In an **electron gun** a filament, heated red hot, gives off electrons, which a positive **anode** pulls across a vacuum. An electron starting at rest and accelerated through a potential difference V gains kinetic energy eV:

$$\frac{1}{2}mv^2 = eV \qquad \Rightarrow \qquad v = \sqrt{\frac{2eV}{m}}$$

So the final speed grows as the **square root** of the voltage: four times the voltage gives twice the speed (Exercise 10). Picture tubes, X-ray tubes and electron microscopes all begin this way.

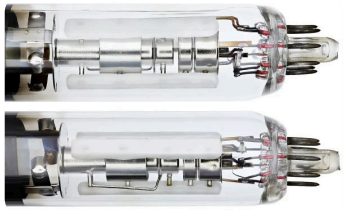


Figure 21.5 *The electron gun of a black-and-white television picture tube made in 1980, seen from two sides. The heated cathode sits in the small cylinder nearest the pins, on the right; the cylinders to its left are held at rising positive potentials, which speed the electrons up and focus them into a beam heading left, towards the screen.*
Photo: Mister rf, CC BY-SA 4.0

WORKED EXAMPLE
An electron through 100 V

An electron starts from rest and is accelerated through 100 V. Find its kinetic energy in electron volts and in joules, and its speed. ($e = 1.60 \times 10^{-19}$ C, $m = 9.11 \times 10^{-31}$ kg)

Answer
Kinetic energy $= eV =$ **100 eV** $= 100 \times 1.60 \times 10^{-19} =$ **1.60×10^{-17} J**.

$$v = \sqrt{\frac{2eV}{m}} = \sqrt{\frac{2 \times 1.60 \times 10^{-17}}{9.11 \times 10^{-31}}} = \sqrt{3.51 \times 10^{13}} = \mathbf{5.93 \times 10^{6}\ m/s}$$

about 2% of the speed of light, after only 100 V.

Check 4
A proton (charge +e) and an alpha particle (charge +2e) are each accelerated from rest through 500 V. Their kinetic energies are:

a both 500 eV
b 500 eV and 1000 eV
c 1000 eV and 500 eV
d 500 eV and 250 eV

21.5 Potential gradients

An electric field can be pictured by its field lines (Unit 20), or by its **equipotentials**: lines, or in three dimensions surfaces, joining points at the same potential. An **equipotential surface** is one on which all points are at the same potential. The potential difference between any two points on it is zero, so **no work is needed to move a charge from one point on it to another**.

A test charge moving along an equipotential has constant potential energy. That can only be so if the field does no work on it, which means the force from the field is at right angles to its motion. So **the field lines cross every equipotential surface at right angles**, along its normal. The equipotentials of an electric field are like those of a gravitational field (Unit 7): both are lines of constant potential energy for a test object, a test charge in one case and a test mass in the other.

The **potential gradient** at a point in an electric field is the change of potential per unit change of distance in a given direction, $\Delta V / \Delta x$.

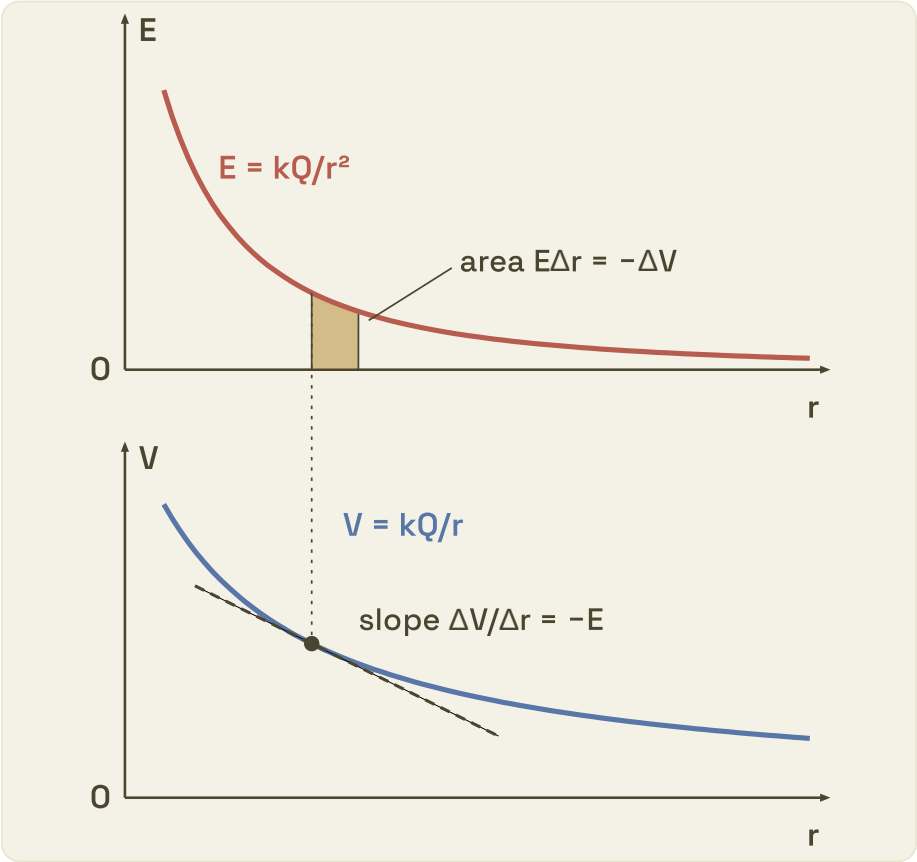


Figure 21.16 *For a positive point charge: top, the field strength $E = kQ/r^2$ against r, with a thin strip of area $E\Delta r$, the fall in potential over Δr; bottom, the potential $V = kQ/r$ against r, with a tangent whose slope is $-E$. The field falls off faster ($1/r^2$) than the potential ($1/r$).*

WORKED EXAMPLE
The potential difference near a point charge

A point charge of +2.0 nC is fixed in air. Find the potential difference between a point 10 cm from it and a point 20 cm from it, and the work needed to carry an electron from the nearer point to the farther one.

Answer

$$V_{10} - V_{20} = kQ\left(\frac{1}{0.10} - \frac{1}{0.20}\right) = 8.99 \times 10^{9} \times 2.0 \times 10^{-9} \times (10 - 5) = 89.9 \approx \mathbf{90\ V}$$

(This is area A of Figure 21.14 (c) for $r_1 = 10$ cm and $r_2 = 20$ cm.)
Carrying the electron outwards takes it to a potential 90 V **lower**: $\Delta V = -90$ V. The work an outside agent must do is $q\,\Delta V = (-1.60 \times 10^{-19})(-90) =$ **$+1.4 \times 10^{-17}$ J** (90 eV): positive, because the positive charge attracts the electron and you must pull it away.

3. Add up the changes from infinity, where $V_\infty = 0$, to a distance r from the charge:

$$\int_\infty^r dV = V_r - V_\infty = V_r = \int_\infty^r -E\,dr = -kQ\int_\infty^r \frac{dr}{r^2} = -kQ\left[-\frac{1}{r}\right]_\infty^r = \frac{kQ}{r}$$

$$\boxed{V = \frac{kQ}{r} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}}$$

This is the same expression as in 21.2, reached from the field instead of from the force. It says that the closer we are to a positive charge, the greater (more positive) the potential; the closer we are to a negative charge, the more negative the potential.

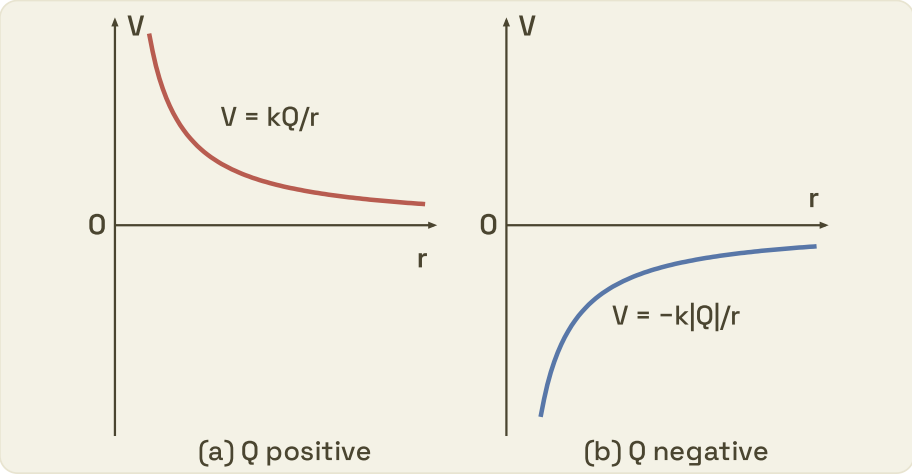


Figure 21.15 *The potential against the distance r from a single point charge Q when the charge is (a) positive and (b) negative. Both approach zero far away.*

Since the potential is the work done in moving a unit positive charge from infinity to a point, the work needed to move a charge q from infinity to that point is $V \times q$. This is the **potential energy** that the charge q has because of its position in the field. If the field comes from a single point charge Q:

$$E_p = qV = \frac{kqQ}{r} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r}$$

Graphs of the field strength and the potential against distance for a single positive point charge show the two descriptions side by side (Figure 21.16): the potential is the area under the field's curve, and the field is minus the slope of the potential's curve.

DID YOU KNOW?

A map you already know

A map with contour lines is a map of gravitational equipotentials: every point on one contour line is at the same height, so a walker moving along it does no work against gravity. Where the lines crowd together, as on the slopes of the hills round the Kathmandu Valley, the ground is steep; where they spread apart, as on the floor of the Kathmandu Valley, it is nearly flat. Water runs downhill at right angles to the contours. Read an electric equipotential map the same way: close lines mean a strong field, and a positive charge is pushed "downhill", across the lines at right angles.

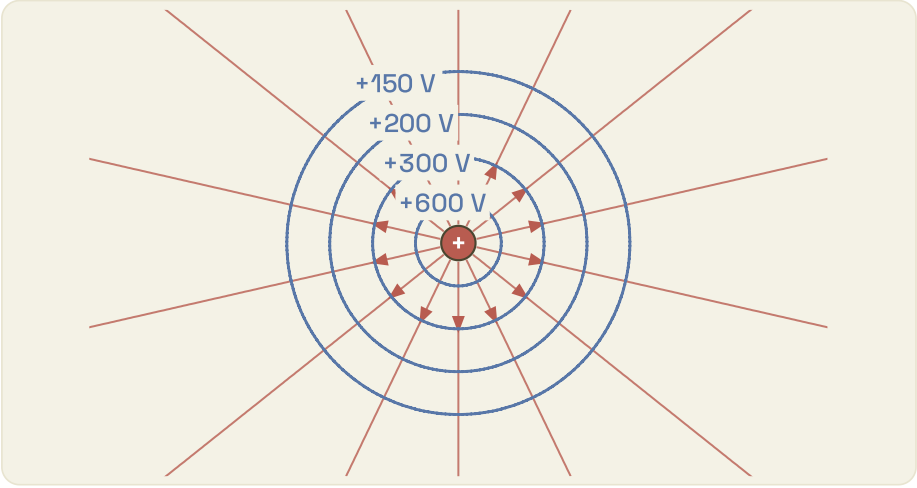


Figure 21.6 *A single positive charge (here +2.0 nC). Field lines (red) go straight out; the equipotentials (blue) are circles, here 3, 6, 9 and 12 cm from it, at 600, 300, 200 and 150 V. Each 3 cm step outwards lowers the potential by less, as the field weakens.*

1. If the field is **non-uniform**, as round the charges above, the potential gradient changes from place to place and with direction. **The closer the equipotentials are, the greater the potential gradient** at right angles to them, and so the stronger the field.
2. If the field is **uniform**, as between two oppositely charged parallel plates, the equipotentials between the plates are **equally spaced** lines parallel to the plates (Figure 21.9).

The graph shows that the potential, measured from the negative plate, is **proportional** to the distance x from it: the potential gradient is the same everywhere between the plates. The potential increases in the opposite direction to the field, and its gradient is $\frac{V}{d}$. The electric field strength between the plates is also $\frac{V}{d}$, directed from the + plate to the − plate. In other words, the electric field strength is the **negative of the potential gradient**:

$$E = \frac{V}{d} \quad \text{(uniform field, from + to −)}$$

The statement still holds when the field is not uniform, but then the gradient has to be taken over a short distance: $\Delta V/\Delta x$, where ΔV is the change of potential between two points a small distance Δx apart, instead of V/d. Because the potential gradient points opposite to the field lines, in general

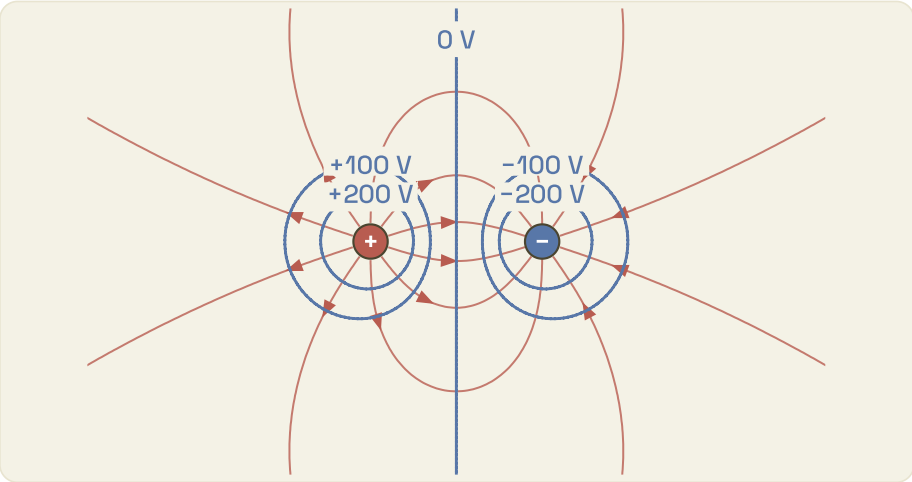


Figure 21.7 Equal and opposite charges (+1.0 nC and −1.0 nC, 12 cm apart). The equipotentials surround each charge, positive round one and negative round the other; the 0 V equipotential is the straight line halfway between them. Every field line crosses every equipotential at a right angle.

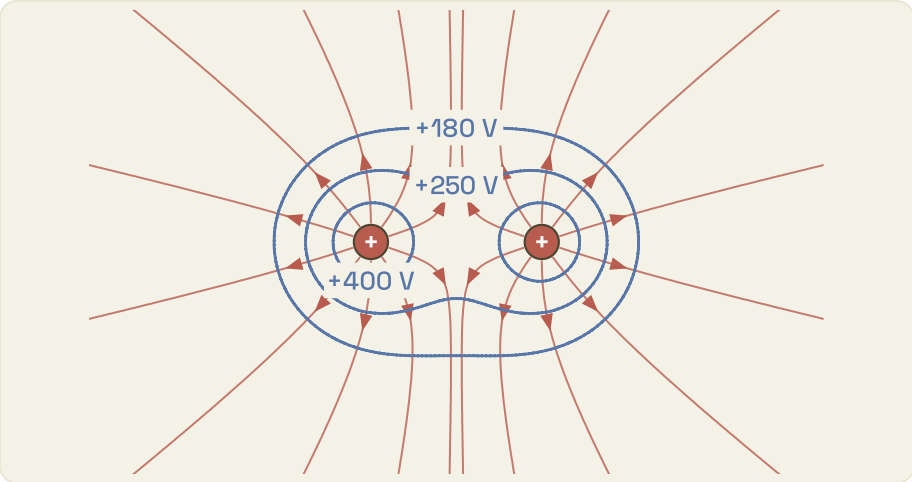


Figure 21.8 Two equal positive charges (+1.0 nC each, 12 cm apart). Close in, each charge has its own equipotentials (400 V); farther out, at 250 V and 180 V, one equipotential surrounds both.

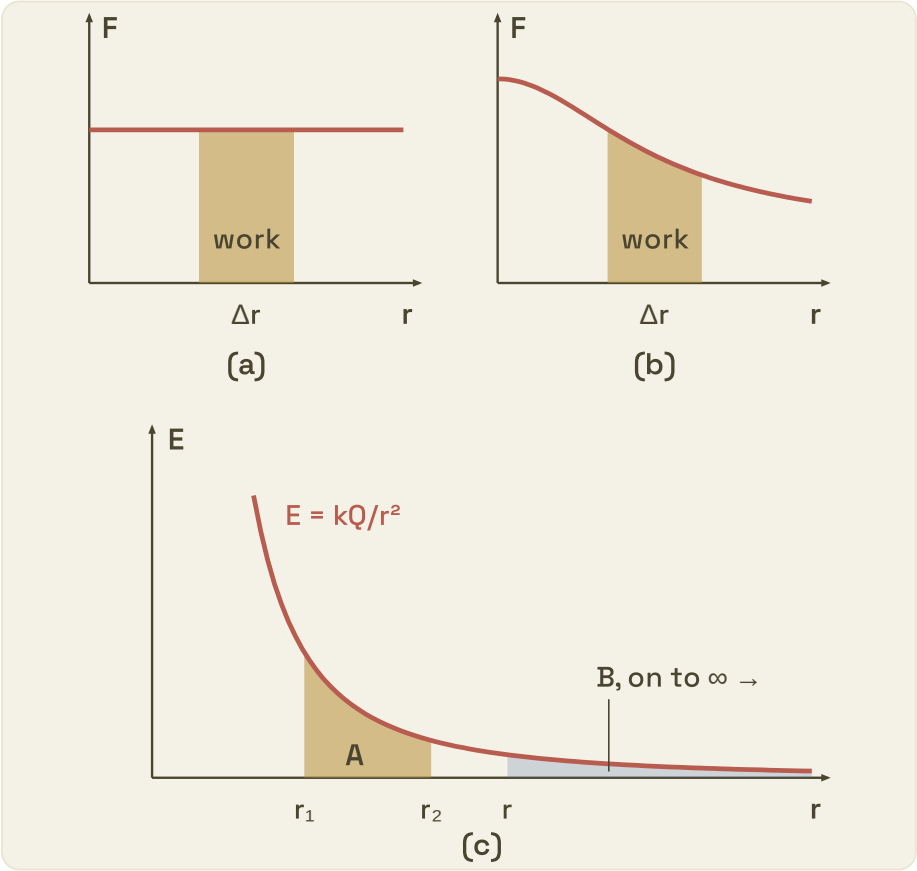


Figure 21.14

a. A constant force: the work is the rectangle $F \times \Delta r$.

b. A varying force: the work is the area under the curve.

c. The field of a positive point charge, $E = kQ/r^2$: area A, between r_1 and r_2, is the potential difference between those two points; area B, from r all the way out to infinity, is the potential at r.

Figure 21.14 (c) shows how the field of a single positive point charge varies with distance: an inverse-square curve. Area A is the potential change between r_1 and r_2. Area B is the potential at r, because it is the change of potential between infinity, where the potential is zero, and r; the area goes on all the way out to infinity, but it is finite, since the curve falls so fast.

Derivation: $V = kQ/r$ from the field

1. From $E = -\dfrac{\Delta V}{\Delta r}$, in the limit of a very small step, $E = -\dfrac{dV}{dr}$, so a small step dr changes the potential by

$$dV = -E\,dr$$

2. For a point charge, $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2} = \dfrac{kQ}{r^2}$.

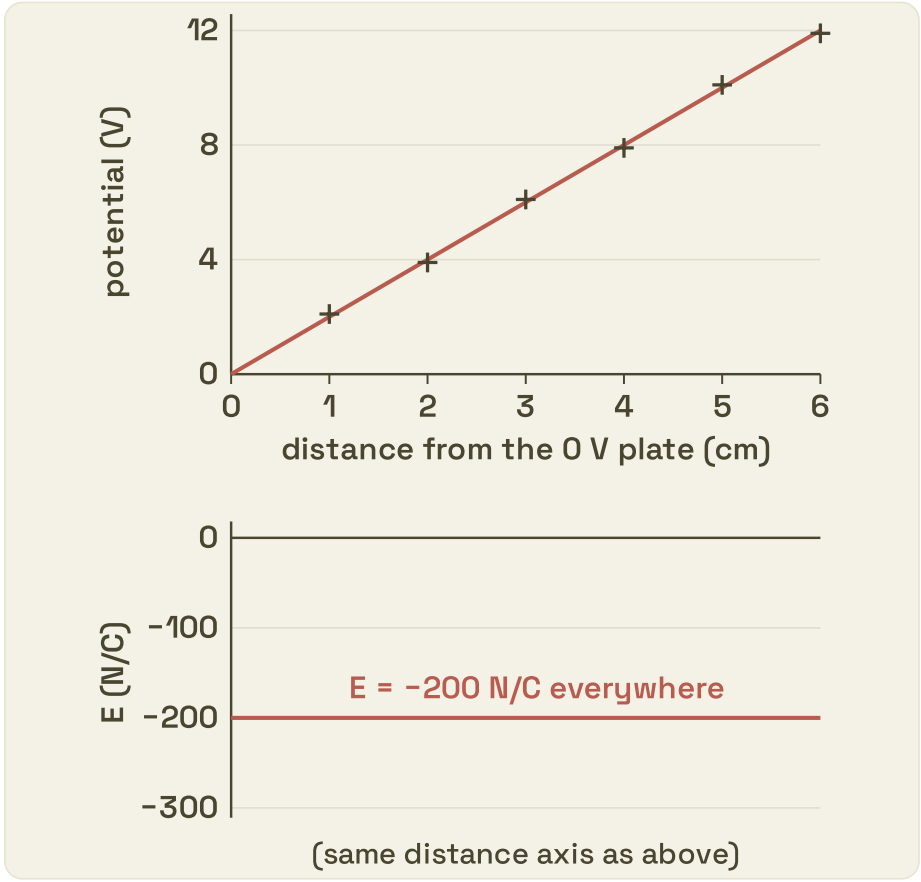


Figure 21.13 *(b) The potential measured at points between the plates against distance from the 0 V plate: the crosses lie on a straight line from 0 V to 12 V. (c) The field strength against distance: −200 N/C everywhere between the plates.*

Check 6

An electron is released from rest at the 0 V plate of Figure 21.12. When it reaches the 12 V plate, it has:

a lost 12 eV of kinetic energy
b gained 12 eV of potential energy
c gained 12 eV of kinetic energy
d gained 12 J of kinetic energy

Areas under the graphs

A graph of force against distance for a **constant** force is a flat line, Figure 21.14 (a). The work done when this force moves an object a distance Δr is the area under the graph, $F \times \Delta r = W$. When the force **varies** with distance, as an electric force usually does, the work done is still the area under the graph, Figure 21.14 (b) (Unit 5, the graphical method).

Electric field strength is force per unit charge, so **the area under a graph of electric field strength against distance is the work done per unit charge: the change in potential**.

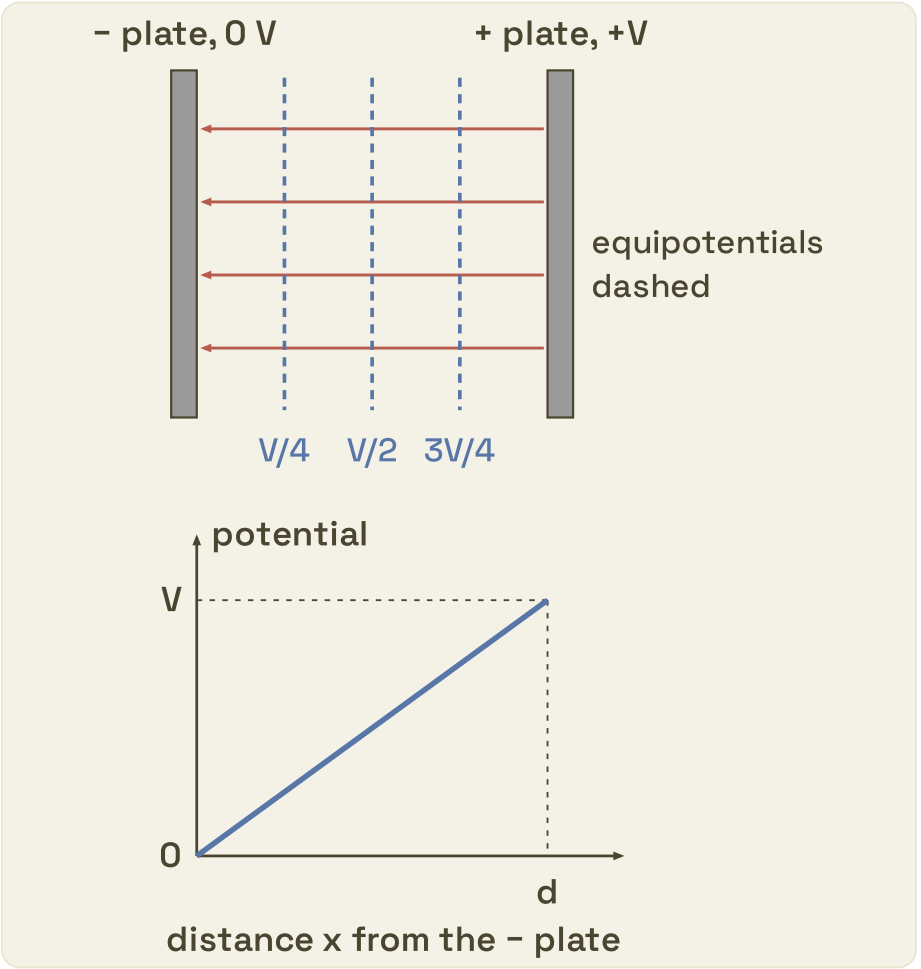


Figure 21.9 *Top: parallel plates a distance d apart, the negative plate at 0 V and the positive plate at +V. The field lines (red) are straight and parallel, from + to −; the equipotentials (dashed) at V/4, V/2 and 3V/4 are equally spaced. Bottom: the potential against the distance x from the negative plate, a straight line from 0 to V.*

Check 5

Two parallel plates 2.0 cm apart are connected to a 240 V supply. A point P is 0.50 cm from the negative plate. The potential at P (taking the negative plate as 0 V) and the field at P are:

a 120 V and 12,000 V/m
b 60 V and 12,000 V/m
c 60 V and 120 V/m
d 180 V and 12,000 V/m



Figure 21.10 *Birds on a power line. Both of a bird's feet are on the same wire, at the same potential, so there is no potential difference across its body and no current through it. A large bird that touches two wires at once, or a wire and an earthed pole, can be killed.*
Photo: Vanja Kovac, CC BY-SA 4.0



Figure 21.11 *A woman touching a Van de Graaff generator at the American Museum of Science and Energy, Oak Ridge, USA. Kept insulated from the ground, her whole body is charged to the dome's potential, which can be hundreds of thousands of volts, yet she feels nothing: her body is one equipotential, so no current flows through it. Her hairs, all charged alike, push each other apart. Touching an earthed object would be another matter.*
Photo: doe-oakridge . 82-25 DOE photo by Frank Hoffman, Public domain

21.6 Graphical interpretations of electric field strength and potential

Figure 21.12 is a reminder of the field between charged parallel plates: the equipotentials are equally spaced. Figure 21.13 (b) is a graph of measurements of the potential at points between the plates, plotted against the distance from the 0 V plate. It is a straight line, because the field is uniform. A graph of the field strength against the distance from the 0 V plate, Figure 21.13 (c), is a straight line parallel to the distance axis.

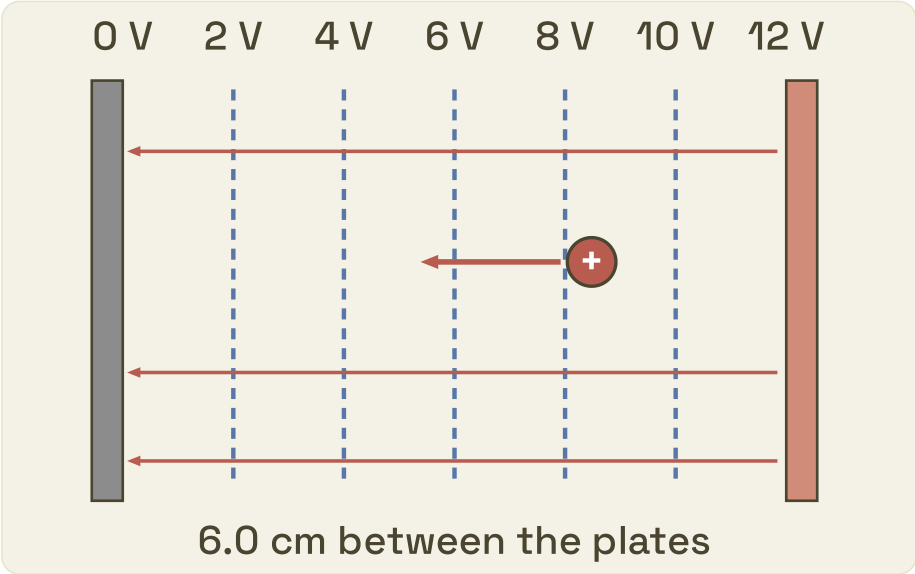


Figure 21.12 *(a) Plates 6.0 cm apart at 0 V and 12 V. The equipotentials, every 2 V, are equally spaced; the field points from the 12 V plate to the 0 V plate, and pushes a positive charge between them towards the 0 V plate.*

The gradient of the potential–distance graph is $\frac{\text{change in } V}{\text{change in } x}$, and since the graph is a straight line, it is $\frac{V}{d} = \frac{12\ \text{V}}{0.060\ \text{m}} = 200\ \text{V/m}$, the size of E. This leads to

$$\text{electric field strength } E = -\text{potential gradient} = -\frac{\Delta V}{\Delta x}$$

(Δ, as usual, stands for "change in".)

What the minus sign means. Here x is measured from the 0 V plate towards the 12 V plate, the direction in which the potential **rises**. The field points the other way, from 12 V back to 0 V, in the negative direction of x. So $E = -\frac{12\ \text{V}}{0.060\ \text{m}} = -200\ \text{V/m}$: a field of 200 V/m pointing towards smaller x. That is why graph (c) is drawn at −200 N/C. The rule always works the same way:

- going **against** the field (upstream) takes you to a **higher** potential: ΔV is positive;
- going **with** the field (downstream) takes you to a **lower** potential: ΔV is negative.

In Figure 21.12 the field pushes the positive charge to the left. Moving left from the 12 V plate to the 0 V plate, $\Delta V = 0 - 12 = -12$ V and $\Delta x = -0.060$ m, so $E = -\frac{\Delta V}{\Delta x} = -\frac{-12}{-0.060} = -200$ V/m, the same answer: whichever way you go, the formula gives the field in the direction of $+x$, and the minus sign of the result says it points the other way.

An electron between the plates. If an electron moves from the 0 V plate towards the 12 V plate, a position of higher potential, its potential energy is **reduced**: $\Delta E_p = q\,\Delta V = (-e)(+12\ \text{V}) = -12$ eV, because the electron's charge is negative and ΔV is positive. If it is free to move it accelerates in this direction, gaining 12 eV of kinetic energy from the field at the expense of its electric potential energy.