Unit 20

Electric Field

This unit online, with its interactives:
science-playground.com/s/11f20

Rub a balloon on your hair and hold it a few centimeters above some torn paper. The paper jumps up to it, though nothing touches it. Before the jump, was anything different in the space between the balloon and the paper? Physics answers yes: the balloon has filled the space round it with an **electric field**, and the paper is pushed by the field where it sits.

That idea, a field filling space, is one of the most useful in all of physics. This unit describes the electric field with pictures (field lines) and with numbers (the field strength), uses it to steer beams of electrons, and ends with **Gauss's law**, a single statement from which the field of a charged sphere, a charged wire or a charged sheet follows in a few lines. Unit 19 gave the force between two charges, and a first look at $E = F/q$; Unit 21 adds the energy.

A STORY FROM SCIENCE

The prince of mathematicians

Carl Friedrich Gauss (1777–1855) was the son of a bricklayer in Brunswick, in Germany. As a schoolboy, the story goes, he was told with the rest of the class to add up the whole numbers from 1 to 100, and handed in the answer, 5050, almost at once: he had paired 1 with 100, 2 with 99 and so on, fifty pairs of 101. A duke who heard of the boy paid for his education, and he received his doctorate from the University of Helmstedt in 1799.

He changed almost every field he touched: the theory of numbers, statistics (the bell-shaped "Gaussian" curve), geometry, and the orbits of planets and comets, when in 1801 he worked out where to look for the lost dwarf planet Ceres. With Wilhelm Weber he built one of the first electric telegraphs, and he founded the Magnetic Union, a network of observatories that measured the Earth's magnetic field at agreed times. The law in 20.6 bears his name, and so did the old unit of magnetic field, the gauss.



Figure 20.1 *Carl Friedrich Gauss (1777–1855), painted in 1840 by Christian Albrecht Jensen.*
Photo: Christian Albrecht Jensen, Public domain

20.1 Introduction

In physics, a **field** is a region in which an object feels a force without being touched. An **electric** (or electrostatic) **field** is made by charged objects, and any other charged object in it feels a force.

You can make an electric field easily by rubbing a glass rod with a silk cloth. The rubbing transfers electrons from the glass to the silk, leaving the cloth negative and the rod positive. The rod is now surrounded by an electric field: it will attract small pieces of paper, or bend a thin stream of water from a tap.

Nature makes its own electric fields. The strong field between the base of a thundercloud and the ground helps to open the path along which the current of a lightning flash flows.

What you need from Unit 19:

1. Electric charge Q is measured in coulombs (C), and can be positive or negative.
2. Opposite charges attract each other; like charges repel.

3. A charged object placed in an electric field feels a force.
4. A uniformly charged sphere acts, outside it, as if all its charge were at its center.
5. Just like gravitational fields, electric fields can be drawn as **field lines**.

20.2 Electric field lines

Sometimes it is obvious where a force comes from: a brake pad rubbing on a bicycle wheel's rim slows it down. In other cases there is no contact at all, yet there is a force: two magnets push each other apart, two charged objects attract. Such forces are said to **act at a distance**.

The word **field** is used for the space round an object in which it exerts such a force. When a comb picks up paper, we say the paper is sitting in the comb's electric field. The idea is powerful because so much is common to all fields, electric, magnetic and gravitational: what you learn here you will use again.

Any two charged objects exert equal and opposite forces on each other without touching. We say an electric field surrounds each charge. To explore a field, imagine a small positive **test charge**: small enough not to disturb the charges that make the field. Wherever you put it, it feels a force in a definite direction. An **electric field line** (or **line of force**) is a line drawn so that at every point on it, its direction (its tangent) is the direction of the force on a positive test charge placed there.

The idea of field lines was introduced by **Michael Faraday** in the 1830s. The lines are imaginary, but they make a field's shape easy to see and to reason about. They are drawn by a few rules:

- Lines **start on positive charges and end on negative charges** (or go off to, or come in from, far away).
- Each line carries an **arrow** showing the direction of the force on a positive charge: away from positive charges, towards negative ones.
- Where the field is **strong**, the lines are **close together**; where it is weak, they are far apart. The lines behave as if they pushed each other apart sideways.
- The lines **never cross**: at a crossing point a test charge would be pushed two ways at once, but the field at a point has only one direction.
- The lines meet the surface of a **conductor at 90°**. (If a line met it at a slant, the field would have a part along the surface, and the conductor's free charges would move until that part was gone.)

> CAREFUL
> **A field line is not the path of a charge**
>
> Some books define a field line as "the path a free positive test charge follows". It isn't, in general. A field line gives the direction of the **force** (and so of the acceleration) at each point, not of the velocity. A moving charge has momentum, so it need not go the way it is pushed: an electron fired across the straight, parallel lines between two plates curves along a parabola (20.4), just as a ball thrown sideways curves while gravity pulls straight down. And where the lines curve, a charge let go from rest starts along a line but then drifts across the lines, to the outside of the bend. A charge follows a field line only when the line is straight and the charge starts from rest on it.

Field patterns

The field patterns above are all traced from Coulomb's law, line by line. Between two parallel plates the field is **uniform** (the same strength and direction everywhere) in the region between them, and weaker towards their edges. At the edges the lines curve outwards, as the field weakens from its large value between the plates to the much weaker field well away from them: these are called **edge effects**.

of the graph, and from it the charge on the nucleus.

ii. The nucleus emits an alpha particle. State the change, if any, to the graph for the resulting (daughter) nucleus. Explain your answer.

Numerical problems

8. A negatively charged droplet of oil is held stationary between two horizontal plates, 2.10 cm apart, with a potential difference of 1.50 kV between them. The droplet is spherical, with a radius of 1.27×10^{-6} m, and the density of the oil is 950 kg/m³. Two forces act on it: its weight, and the electric force.

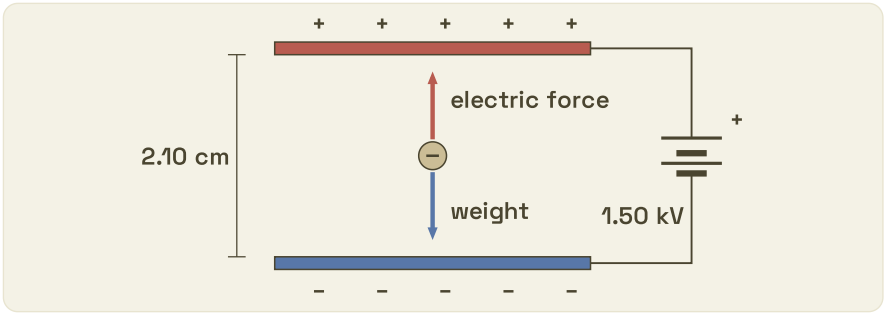


Figure 20.24 The oil drop between the plates, and the two forces on it.

i. Show that the size of the charge on the droplet is about 1.1×10^{-18} C. ($g = 9.8$ m/s²)

ii. Calculate the number of electrons that make up this charge.

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

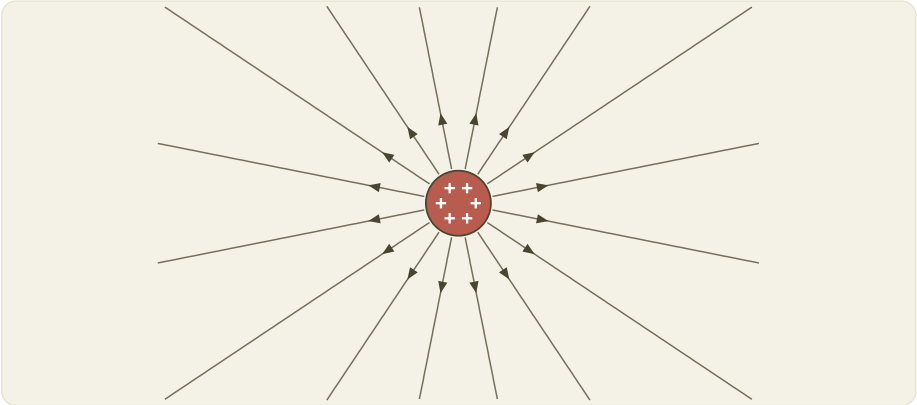


Figure 20.2 A charged sphere: the lines run straight out from it, like spokes, as from a point charge at its center.

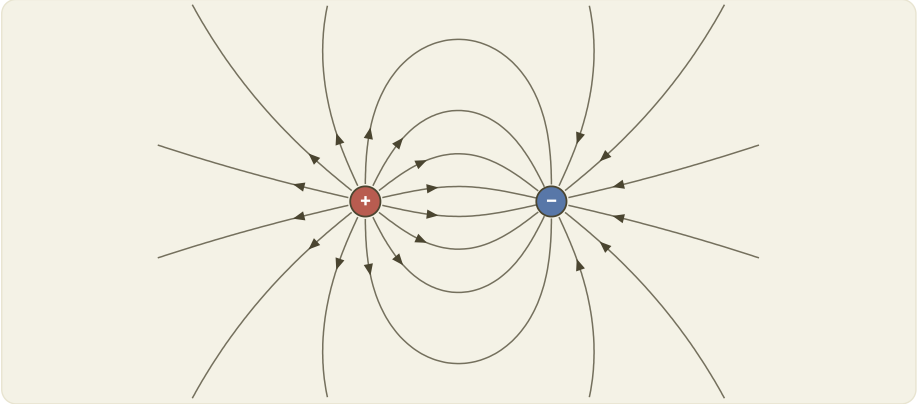


Figure 20.3 A positive and a negative charge: every line from the + ends on the −. A pair of equal and opposite charges is called a **dipole**.

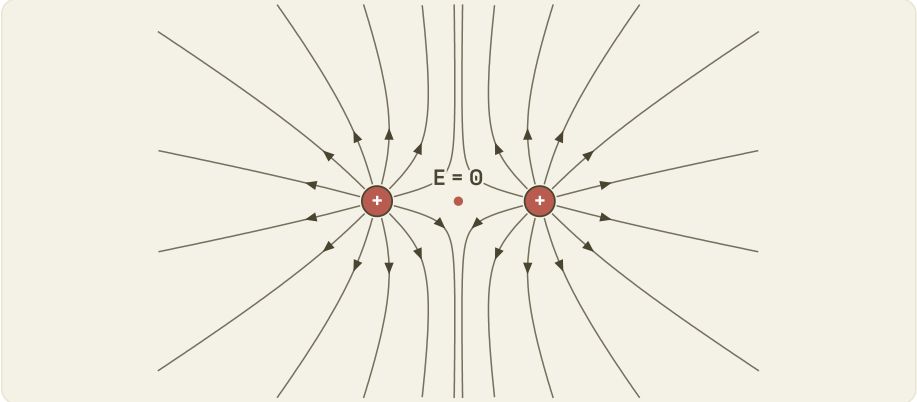


Figure 20.4 Two equal positive charges: the lines bend away from each other. Midway between them is a **neutral point**, where the two fields cancel.

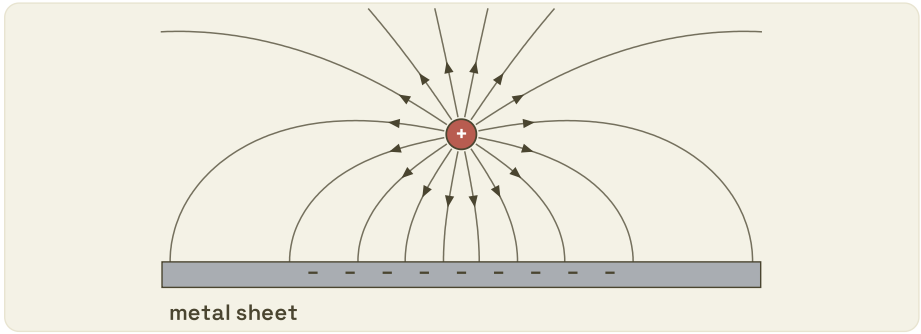


Figure 20.5 *A positive charge near a flat metal sheet: the lines curve round to meet the metal at right angles, ending on negative charges that the charge has drawn to the sheet's surface by induction.*

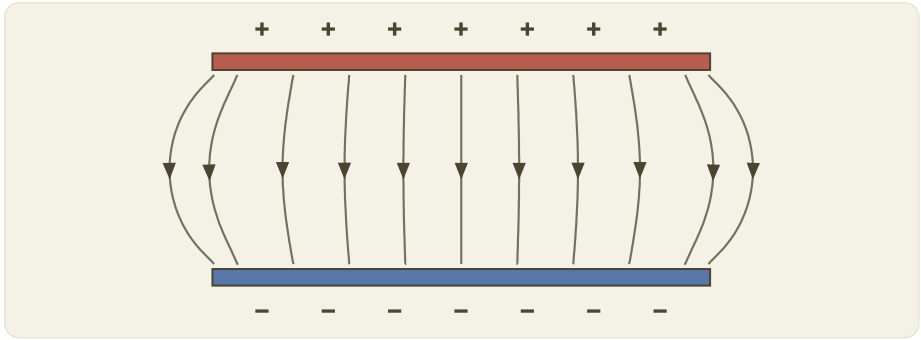


Figure 20.6 *Two long parallel plates with opposite charges, in cross-section: between them the lines are straight, parallel and evenly spaced (a uniform field); at the ends they bow outwards.*

Radial fields

The field between two plates is uniform. What shape of field does a small "point" charge, such as a single proton, make? An isolated positive charge makes the field of Figure 20.2: the lines go straight out from it in every direction, so it is called a **radial field**. The lines spread out in three dimensions from the charge, like the spines of a hedgehog; the drawing shows only a flat slice through it. For a negative charge, the arrows point inwards.

The **equipotential surfaces** round a point charge (Unit 21) are spheres centered on it, drawn as circles in a slice.

A charged sphere, or even a roughly round charged head, also makes a radial field, as the hair in the photo shows.

Check 7

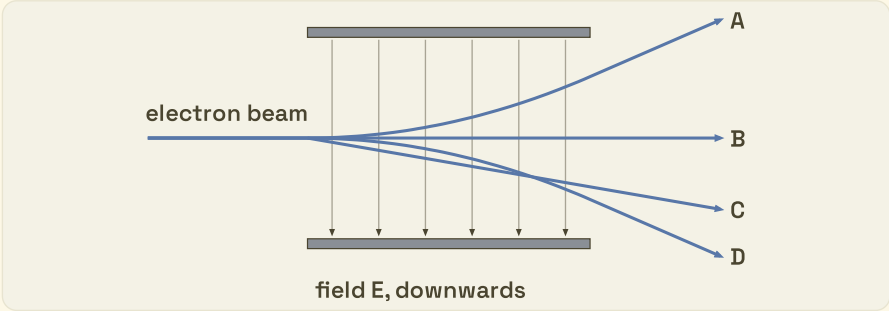


Figure 20.20 *For question 1: an electron beam and four possible paths through a field pointing down.*

1. Which path in Figure 20.20 shows a possible movement of an electron in the electric field shown?

a A
b B
c C
d D

Check 8

2. Which describes the electric field pattern of an isolated negative point charge?

a straight lines radiating out from the charge, arrows pointing away from it
b straight lines radiating out from the charge, arrows pointing towards it
c circles round the charge, arrows going clockwise
d circles round the charge, arrows going anticlockwise

Check 9

3. Which one of the following correctly describes the electric field strength?

a It is defined as charge per unit force.
b It is defined as force per unit charge.
c It is defined as the energy per unit charge.
d It is defined as the charge per unit energy.

Check 10

4. The electric field between two charged plates is 650 N/C. If the distance between the plates is 12 cm, what should be the potential difference between them?

a 54 V
b 78 V
c 4500 V
d 7800 V

Check 11

5. Two horizontal conducting plates are 5 cm apart. The upper plate is at +10 V and the lower plate at 0 V. Which describes the electric field between the plates?

a 2 N/C, pointing up
b 2 N/C, pointing down
c 200 N/C, pointing up
d 200 N/C, pointing down

2. **Choose the Gaussian surface**: a short cylinder through the sheet, its two flat ends (each of area A) parallel to the sheet, one on each side.
3. **The curved side**: the field runs along it, so no flux.
4. **The two ends**: the field crosses each at right angles, outwards: flux EA through each, $2EA$ in all.
5. **The charge enclosed**: the patch of sheet inside the cylinder, $q_{in} = \sigma A$.
6. **Gauss's law**: $2EA = \dfrac{\sigma A}{\varepsilon_0}$, so

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

The field **doesn't depend on the distance** from the sheet at all: a large flat sheet makes a uniform field. Two parallel plates with opposite charges $+\sigma$ and $-\sigma$ add their fields between them (each $\sigma/2\varepsilon_0$, both pointing from + to −) and cancel outside, so between the plates

$$E = \frac{\sigma}{\varepsilon_0}$$

uniform, as in 20.4: Gauss's law explains why the field between parallel plates is uniform.

Check 6

At 10 cm from a long charged wire the field is 6.0×10^4 N/C. At 30 cm from it, the field is:

a 6.7×10^3 N/C
b 2.0×10^4 N/C
c 1.8×10^5 N/C
d 6.0×10^4 N/C

IN SHORT

Key ideas of Unit 20

- An **electric field** is the region round charges in which another charge feels a force. **Field lines** show the direction of the force on a + charge at each point (not the path of a moving charge), start on + and end on −, never cross, crowd together where the field is strong, and meet conductors at 90°.
- **Field strength** $E = F/q$, a vector, in N/C or V/m; the force on any charge is $F = qE$. For a point charge $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}$, a radial field.
- Between parallel plates the field is uniform, $E = V/d$. A charge crossing it follows a parabola, $y = \dfrac{qE}{2mv^2}x^2$; one moving along it gains $\frac{1}{2}mv^2 = qV$.
- **Electric flux** $\Phi = EA\cos\theta$ (N m²/C), θ between the field and the surface's normal.
- **Gauss's law**: the flux out of any closed surface is $\Phi = q/\varepsilon_0$, q the net charge inside.
- Charged sphere: $E = \dfrac{Q}{4\pi\varepsilon_0 r^2}$ outside, $E = \dfrac{Qr}{4\pi\varepsilon_0 a^3}$ inside (insulating), greatest at the surface. Line: $E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$. Sheet: $E = \dfrac{\sigma}{2\varepsilon_0}$; between two plates $E = \sigma/\varepsilon_0$.

Exercise

Choose the correct option

Answer these questions

6. The figure shows the electric field pattern drawn by a student for two oppositely charged plates set at right angles. State two errors the student has made.

Check 1



Figure 20.7 *At Science City in Kolkata, a girl touches the dome of a Van de Graaff generator, a machine that builds up a large charge. The charge spreads over her too; each hair, charged alike, is pushed away from her head and from the others, and points along the radial field.*
Photo: Biswarup Ganguly, CC BY 3.0

Which is NOT a rule for drawing electric field lines?

a They point away from positive charges and towards negative ones.
b They are closer together where the field is stronger.
c They can cross where two fields overlap.
d They meet a conductor's surface at right angles.

20.3 Electric field strength

The **electric field strength** E at a point in a field is the **force per unit charge on a small positive test charge** placed at that point.

Imagine an isolated charge Q in space, and a point P a distance r from it. Put a small positive test charge q at P and measure the force F on it. Then

$$E = \frac{F}{q}$$

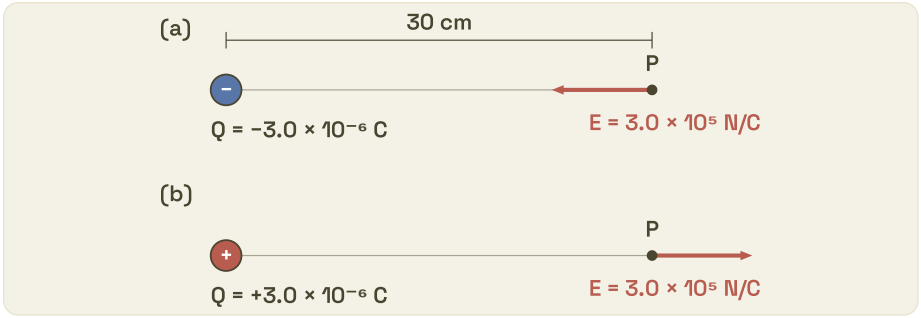


Figure 20.8 *The field at a point P, 30 cm from a point charge: towards a negative charge (a), away from a positive one (b).*

- The SI unit of E is the **newton per coulomb**, N/C. The **volt per meter**, V/m, means the same thing (20.4 and Unit 21 show why).
- E is a **vector**, in the direction of the force on a *positive* test charge: dividing a vector by a positive number q only scales it up or down. The force on a negative charge points the other way, against the field.
- The test charge must be small, so that it doesn't push the charges making the field out of place. E then doesn't depend on q: double the test charge and the force doubles too.

Once E at a point is known, the force on **any** charge q put there is

$$F = qE$$

Derivation: the field of a point charge

1. By Coulomb's law, the force on the test charge q a distance r from Q is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}$$

2. The field strength is the force per unit charge:

$$E = \frac{F}{q} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2} \times \frac{1}{q}$$

3. The test charge cancels:

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = \frac{kQ}{r^2}}$$

The field depends only on the charge that makes it and the distance, not on the test charge. Measuring r from the charge out to the point, a **positive** E (from a positive Q) points **outwards**, away from the charge, which is what we expect when both it and the test charge are positive. When Q is **negative**, E points **towards** Q. The lines of a point charge radiate straight out (positive) or in (negative): a radial field.

The field falls off as $1/r^2$: twice as far, a quarter as strong.

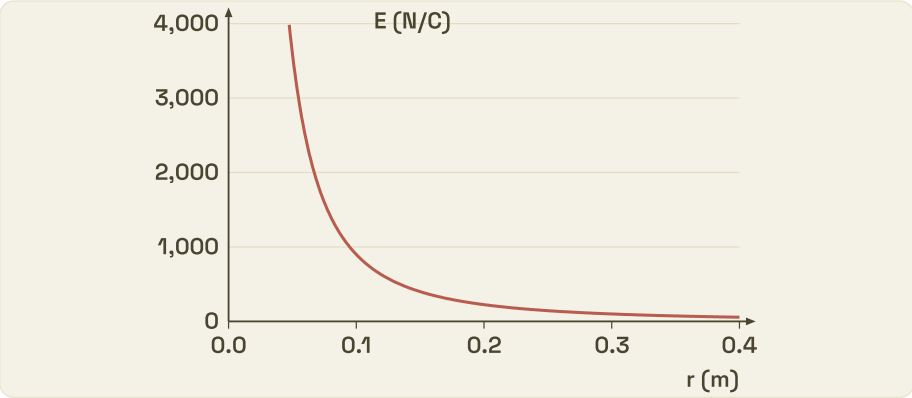


Figure 20.9 *The field strength of a 1.0 nC point charge against distance, drawn from $E = kQ/r^2$. Halving the distance from 0.2 m to 0.1 m makes the field four times as strong.*

WORKED EXAMPLE

The field that holds an atom together

Find the electric field of a hydrogen atom's proton at the electron's average distance, 0.53×10^{-10} m.

Answer

$$E = \frac{kQ}{r^2} = \frac{9.0 \times 10^9 \times 1.6 \times 10^{-19}}{(0.53 \times 10^{-10})^2} = 5.1 \times 10^{11}\ \text{N/C}$$

pointing away from the proton. Ordinary air breaks down and sparks at about 3×10^6 N/C, so inside every atom is a field over a hundred thousand times stronger than any spark you have seen.

Check 2

A charge of $+2.0$ μC placed at a point feels a force of 0.040 N to the east. A charge of -4.0 μC placed at the same point instead would feel:

a 0.040 N to the west
b 0.080 N to the east
c 0.080 N to the west
d 0.020 N to the west

The field between two parallel plates

Join two flat metal plates, facing each other, to the terminals of a battery. One plate becomes positive, the other negative, and between them is the field of Figure 20.6, drawn again with its battery below. The field lines are **parallel** to each other, at **right angles to the plates**, run **from the positive plate to the negative**, and are evenly spaced: the field is **uniform**.

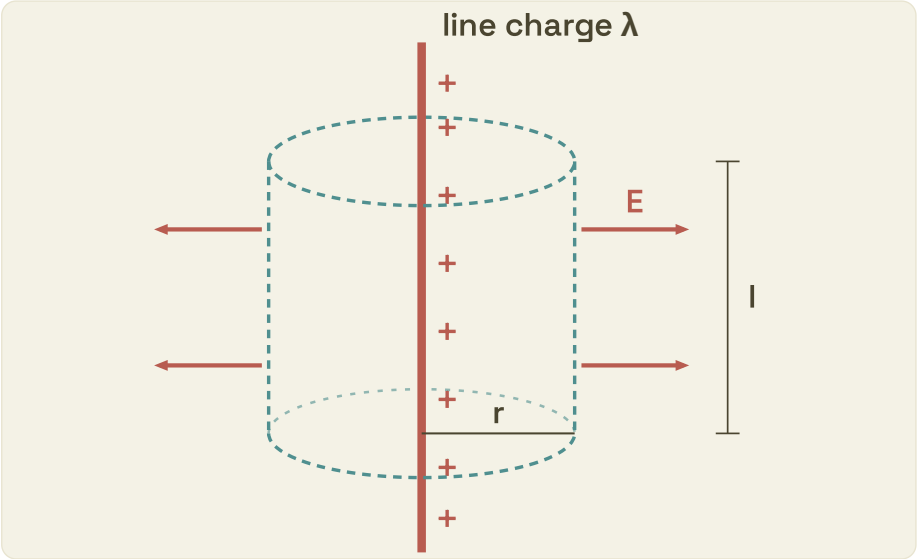


Figure 20.18 *A long line of charge with a coaxial cylindrical Gaussian surface of radius r and length l. The field points straight out from the line, through the curved side only.*

7. The length l cancels (as it must: we chose it):

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

The field of a line falls off as $1/r$, not $1/r^2$: its lines spread out in only two directions, round the line, instead of three. (It holds for a real wire at distances much smaller than its length, away from its ends.)

An infinite sheet of charge

The same reasoning gives the field of a large flat sheet with charge spread evenly over it, σ coulombs per square meter (the **surface charge density**).

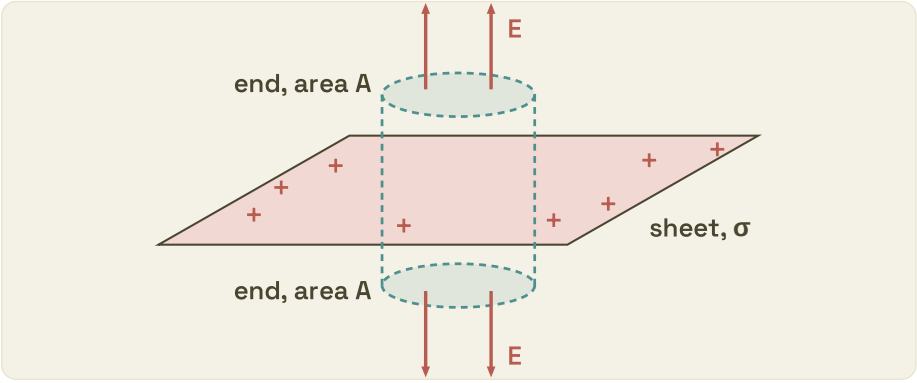


Figure 20.19 *A large sheet of charge with a short cylindrical Gaussian surface (a "pillbox") through it. The field points straight away from both faces of the sheet, through the two ends only.*

1. **Symmetry.** A positive sheet has no reason to push one way along itself rather than another, so the field points straight away from it, at right angles, on both sides, with the same size at the same distance on either side.

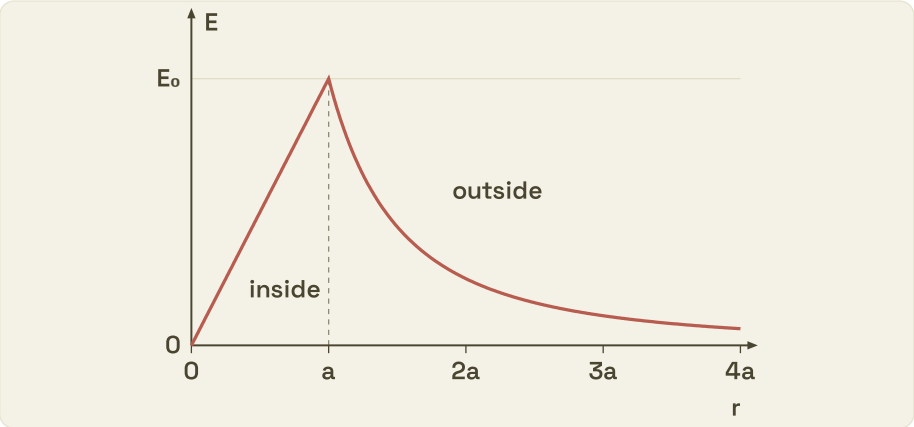


Figure 20.17 *The field of a uniformly charged insulating sphere: a straight line from zero at the center to its greatest value E_0 at the surface, then falling as $1/r^2$.*

WORKED EXAMPLE

A charged ball

An insulating ball of radius 10 cm carries 2.0 μC spread evenly through it. Find the field 5.0 cm, 10 cm and 20 cm from its center.

Answer

With $k = 8.99 \times 10^9$ N m²/C², $Q = 2.0 \times 10^{-6}$ C, $a = 0.10$ m:

- at 5.0 cm (inside): $E = \dfrac{kQr}{a^3} = \dfrac{8.99 \times 10^9 \times 2.0 \times 10^{-6} \times 0.050}{(0.10)^3} = 9.0 \times 10^5$ N/C;
- at 10 cm (the surface): $E = \dfrac{kQ}{a^2} = \dfrac{1.80 \times 10^4}{0.010} = 1.8 \times 10^6$ N/C;
- at 20 cm (outside): $E = \dfrac{kQ}{r^2} = \dfrac{1.80 \times 10^4}{0.040} = 4.5 \times 10^5$ N/C.

Half way to the center, half the surface field; twice as far from the center, a quarter of it.

DID YOU KNOW?

Inside a metal, no field at all

A **conducting** sphere behaves differently from an insulating one. Its free charges push one another as far apart as they can, all onto its outer surface. A Gaussian surface just inside the metal then encloses no charge, so the field inside is **zero** everywhere, and outside it is again kQ/r^2. This is why a car or an airplane hit by lightning keeps the people inside safe: the charge runs over the metal skin, and the field inside stays nearly zero. A metal box used this way is called a **Faraday cage**.

20.9 Gauss's law for an infinite line of charge

A very long, straight, thin wire carries charge spread evenly along it, λ coulombs per meter (the **linear charge density**).

1. **Symmetry.** The line is infinitely long and the same all along, so the field is the same at all points the same distance from it, whatever their height along the line. And it can only point straight out from the line, at right angles to it.
2. **Choose the Gaussian surface**: a cylinder of radius r and length l, with the line along its axis.
3. **The ends**: the field runs along them, parallel to their surfaces, so the flux through both ends is zero.
4. **The curved side**: the field is at right angles to it and the same size all over it. Its area is $A = 2\pi rl$, so the flux is $E \times 2\pi rl$.
5. **The charge enclosed** is the charge on length l of the line: $q_{in} = \lambda l$.
6. **Gauss's law**:

$$E\,(2\pi rl) = \frac{\lambda l}{\varepsilon_0}$$

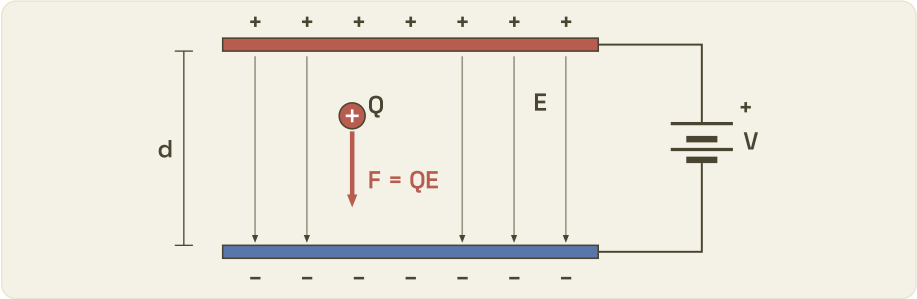


Figure 20.10 *Two parallel plates d apart, joined to a battery of p.d. V. The field between them is uniform, so a charge Q feels the same force F = QE wherever it is between them.*

20.4 The field between the plates is uniform

"Uniform" means the field strength has the **same size and direction everywhere** between the plates (away from their edges). Its size can be found from the **potential difference** V between the plates and their separation d:

$$\boxed{E = \frac{V}{d}}$$

Derivation: $E = V/d$

Consider a small positive charge Q between the plates (Figure 20.10).

1. The force on it is $F = QE$, where E is the field strength between the plates. It is the same everywhere between them, because the field is uniform.
2. If the charge moves from the positive plate to the negative plate, a distance d along the force, the work done on it by the field is force × distance moved:

$$W = Fd = QEd$$

3. By definition, the potential difference between the plates is the work done per unit charge when a charge moves from one plate to the other:

$$V = \frac{W}{Q} = \frac{QEd}{Q} = Ed$$

4. Rearranging $V = Ed$:

$$E = \frac{V}{d}$$

This is why the volt per meter is a unit of field strength: 1 V/m = 1 (J/C)/m = 1 N m/(C m) = 1 N/C.

WORKED EXAMPLE

A battery and two plates

Two parallel plates 4.0 mm apart are joined to a 12 V battery. Find the field between them, and the force on an electron there.

Answer

$$E = \frac{V}{d} = \frac{12}{4.0 \times 10^{-3}} = 3.0 \times 10^3 \text{ V/m}$$

from the positive plate towards the negative one.

$$F = eE = 1.6 \times 10^{-19} \times 3.0 \times 10^3 = 4.8 \times 10^{-16} \text{ N}$$

towards the **positive** plate, against the field, since the electron is negative. Tiny as it is, it would give the electron an acceleration of $F/m = 4.8 \times 10^{-16}/9.11 \times 10^{-31} = 5.3 \times 10^{14}$ m/s², fifty trillion times g.

Charges moving along the field

A charge q of mass m let go from rest at one plate is pushed straight across by the constant force qE: it speeds up with constant acceleration $a = qE/m$, just as a stone dropped in uniform gravity does. By the time it reaches the other plate the field has done work $qEd = qV$ on it, so its kinetic energy is

$$\tfrac{1}{2}mv^2 = qV$$

This is how every electron gun works, from an old TV's picture tube to an X-ray machine in a hospital: electrons boiled off a hot wire are accelerated through a p.d. of thousands of volts. Unit 21 takes it further.

Deflecting a beam of charged particles

A beam of charged particles fired **across** the field between two plates is bent sideways. By changing the p.d. between the plates, you control how much.

Suppose a beam of identical positively charged particles enters the field between plates A (positive) and B (negative), moving parallel to the plates. They are deflected towards the negative plate B; negatively charged particles would be deflected towards the positive plate. The field is uniform, so the force on each particle is constant in size and direction, and the beam curves along a **parabola**, exactly as a ball thrown horizontally does under gravity (Unit 3). On a screen past the plates, the beam's path can be seen and measured.

If the p.d. between the plates is V_P and their separation is d, the field is $E = V_P/d$, and the force on each particle of charge q is

$$F = qE = \frac{qV_P}{d}$$

Derivation: the path is a parabola

Take the particle entering the field at the origin with speed v, parallel to the plates, x along the plates and y across them, towards the plate it is pushed to. Let each plate be L long.

1. **Across the plates** the force is constant, so the acceleration is constant:

$$a = \frac{F}{m} = \frac{qE}{m} = \frac{qV_P}{md}$$

where m is the particle's mass.

2. **Along the plates** there is no force, so the particle keeps its speed v (its starting velocity is the whole of the horizontal part). After a time t:

$$x = vt \quad \text{(i)}$$

3. Across the plates it starts with no velocity, so after the time t:

$$y = \tfrac{1}{2}at^2 = \tfrac{1}{2}\,\frac{qE}{m}\,t^2 \quad \text{(ii)}$$

4. Eliminate t: from (i), $t = x/v$. Put it into (ii):

$$\boxed{y = \frac{qE}{2mv^2}\,x^2}$$

This is $y = (\text{constant}) \times x^2$: **the path is a parabola**.

5. The particle is between the plates for a time $t = L/v$, so it leaves them deflected by

$$y = \frac{qE}{2mv^2}\,L^2 = \frac{qV_P L^2}{2mdv^2}$$

The deflection is **directly proportional to the plate p.d.** V_P: double the p.d., double the deflection. That is what makes the deflecting plates a measuring instrument, as in the cathode ray oscilloscope.

For an **electron**, $q = e$ and $m = m_e$:

$$y = \frac{eE}{2m_e v^2}\,x^2$$

Past the plates there is no force, and the particle goes straight on along the tangent to the parabola where it left.

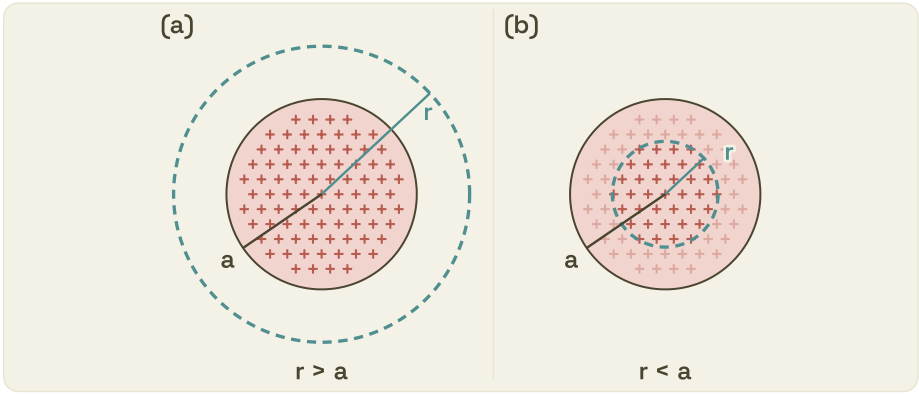


Figure 20.16 *A uniformly charged insulating sphere of radius a, and a spherical Gaussian surface of radius r: (a) outside the sphere; (b) inside it.*

The field at a point outside the sphere ($r > a$)

1. The Gaussian surface of radius r (Figure 20.16a) encloses all the charge, Q.
2. By Gauss's law, $\Phi = EA = \dfrac{Q}{\varepsilon_0}$, with $A = 4\pi r^2$.
3. So $E \times 4\pi r^2 = \dfrac{Q}{\varepsilon_0}$, and

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{r^2}} \quad (r > a)$$

Outside it, the field of a uniformly charged sphere is exactly that of a point charge Q at its center. This is the proof promised in Unit 19, and it holds for gravity too: it is why a planet pulls as if all its mass were at its center (Unit 7).

The field at a point inside the sphere ($r < a$)

1. Now the Gaussian surface of radius $r < a$ (Figure 20.16b) encloses only part of the charge. The volume inside it is $V' = \frac{4}{3}\pi r^3$, so the charge inside is

$$q_{in} = \rho V' = \rho \times \tfrac{4}{3}\pi r^3$$

2. By Gauss's law, $E \times 4\pi r^2 = \dfrac{q_{in}}{\varepsilon_0} = \dfrac{\rho \times \frac{4}{3}\pi r^3}{\varepsilon_0}$
3. Dividing by $4\pi r^2$:

$$E = \frac{\rho}{3\varepsilon_0}\,r$$

4. The whole sphere, of volume $\frac{4}{3}\pi a^3$, holds Q, so $\rho = \dfrac{Q}{\frac{4}{3}\pi a^3}$. Putting it in:

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{a^3}\,r} \quad (r < a)$$

Inside, the field grows **in proportion to** r: it is zero at the center (where the pushes of the charge all round cancel) and grows steadily outwards, because the charge enclosed ($\propto r^3$) grows faster than the area ($\propto r^2$).

The field at the surface ($r = a$)

Both formulas give the same value at $r = a$:

$$E = \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{a^3}\,a = \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{a^2}$$

The field is greatest at the surface and the graph has no jump there.

croprocessor of your phone, the flux of the electric field through any closed surface depends only on the charge it encloses. In nearly 200 years of experiments, no electric field has ever been found that breaks Gauss's law.

Check 5

A closed surface encloses charges of +5 nC, −2 nC and +1 nC, and a charge of +10 nC lies just outside it. The total flux through the surface is:

a 14 nC/ε_0 b 8 nC/ε_0
c 4 nC/ε_0 d 0

20.7 Gauss's law for a point charge

Check the law on the simplest case: a point charge q, surrounded by a sphere of radius r centered on it.

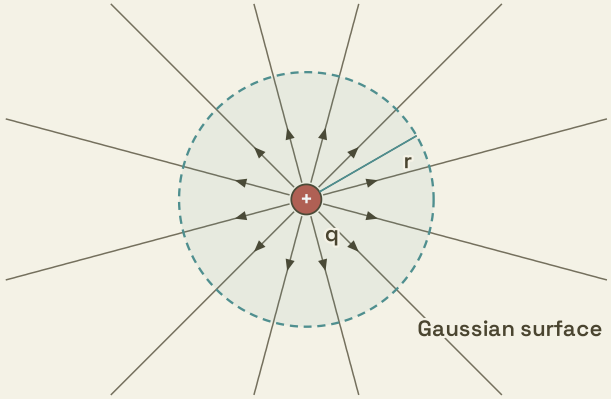


Figure 20.15 *A point charge q at the center of a spherical Gaussian surface of radius r. The field crosses the surface at right angles everywhere, with the same strength.*

1. On the sphere, the field has the same size everywhere, since every point of it is the same distance r from the charge:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

2. The field points straight out, so it is at right angles to the surface (along its normal) at every point: $\cos\theta = 1$.
3. So the flux is just E times the whole area of the sphere, $A = 4\pi r^2$:

$$\Phi = EA = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \times 4\pi r^2 = \frac{q}{\varepsilon_0}$$

The flux **doesn't depend on the radius** r: a bigger sphere has a weaker field on it, $1/r^2$, but a larger area, r^2, and the two cancel exactly. That is why Gauss's law works for every surface: it is the inverse square law in disguise. It is also why Coulomb's constant is written with a 4π: the area of a sphere.

The argument also runs backwards. Start from Gauss's law and the symmetry (a point charge has no favorite direction, so its field must point straight out and have the same size all over the sphere): then $E \times 4\pi r^2 = q/\varepsilon_0$, which **is** Coulomb's law. Gauss's law and Coulomb's law say the same thing.

20.8 Gauss's law: a uniformly charged sphere

An **insulating** solid sphere of radius a carries a total positive charge Q, spread evenly through its volume, with a **volume charge density** ρ (charge per unit volume).

The charge has spherical symmetry, so the field must point straight out from the center and have the same size at every point at the same distance r from it. So choose spherical Gaussian surfaces centered on the sphere: on each, E is the same everywhere and at right angles to the surface, and the flux is simply $E \times 4\pi r^2$.

WORKED EXAMPLE

Deflecting an electron

An electron moving at 2.0×10^7 m/s enters, parallel to the plates, the field between two plates 5.0 cm long and 2.0 cm apart, with a p.d. of 100 V between them. By how much is it deflected by the time it leaves the plates?

Answer

$$E = \frac{V_P}{d} = \frac{100}{0.020} = 5.0 \times 10^3 \text{ V/m}$$

$$y = \frac{eEL^2}{2m_e v^2} = \frac{1.6 \times 10^{-19} \times 5.0 \times 10^3 \times (0.050)^2}{2 \times 9.11 \times 10^{-31} \times (2.0 \times 10^7)^2} = \frac{2.0 \times 10^{-18}}{7.29 \times 10^{-16}} = 2.7 \times 10^{-3} \text{ m}$$

About **2.7 mm**, towards the positive plate. (It takes only $L/v = 2.5 \times 10^{-9}$ s, 2.5 nanoseconds, to cross.) Check it in the widget above.

CAREFUL

Gravity doesn't matter here

Why ignore the electron's weight? Its weight is $m_e g = 9.11 \times 10^{-31} \times 9.8 = 8.9 \times 10^{-30}$ N, while the electric force in this example is $eE = 8.0 \times 10^{-16}$ N, about 10^{14} times bigger. For electrons, protons and ions in a lab's electric fields, gravity can always be left out. For an oil drop (exercise question 8) it can't.

Check 3

A beam of protons and a beam of electrons enter the same field between two plates, both with the same speed, moving parallel to the plates. Compared with the electrons, the protons are deflected:

a by the same amount, the other way b by more, the same way
c by much less, the other way d by much less, the same way

20.5 Electric flux

Field lines have so far been a picture. Now we count them. Consider a field that is **uniform** in size and direction, and a flat surface of area A at **right angles** to it (Figure 20.11a).

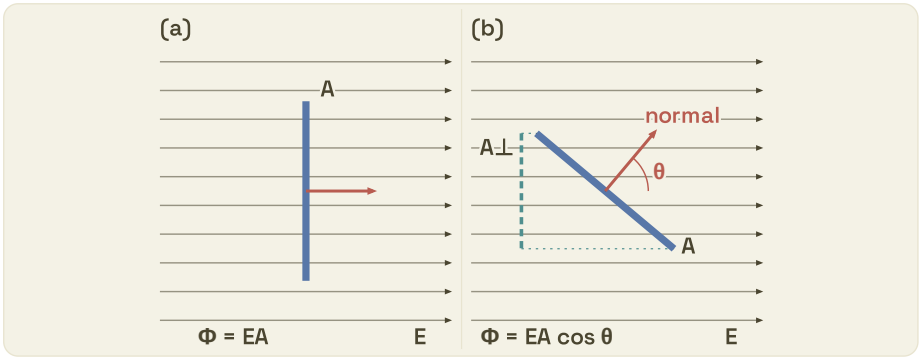


Figure 20.11 *A flat surface of area A in a uniform field, seen edge on. (a) At right angles to the field: Φ = EA. (b) Tilted, its normal at θ to the field: the lines through it are those through its projection A⊥ = A cos θ, so Φ = EA cos θ.*

The number of lines through each square meter (the line density) is drawn in proportion to the field strength. So the total number of lines through the surface is proportional to $E \times A$. This product of the field strength and the area at right angles to the field is called the **electric flux**, Φ (the capital Greek letter phi):

$$\Phi = EA$$

From the units of E and A, flux is measured in newton meters squared per coulomb, **N m²/C**. Electric flux is a measure of the number of field lines passing through a surface.

If the surface is **not** at right angles to the field, fewer lines pass through it. Let the **normal** to the surface (the line at right angles to it) make an angle θ with the field (Figure 20.11b). The lines that pass through A are exactly those that pass through $A_\perp$, the

projection of A on a plane at right angles to the field. From the drawing,

$$A_\perp = A\cos\theta$$

so the flux through the tilted surface is

$$\boxed{\Phi = EA_\perp = EA\cos\theta}$$

- The flux through a surface of fixed area is **greatest**, EA, when the surface is at right angles to the field (its normal along the field, $\theta = 0^\circ$).
- The flux is **zero** when the surface lies along the field (its normal at right angles to the field, $\theta = 90^\circ$): the lines skim past without crossing it.

For a **closed** surface (one with an inside and an outside, such as a box or a ball), the normal is taken pointing **outwards**: flux out of the surface counts as positive, flux into it as negative.

YOUR TURN

Test yourself: flux through a cube

Figure 20.12 shows a cube of side s in a uniform field E. What is the flux through each of the three faces A, B and C with the cube placed as in (a)? Repeat for (b), with the cube turned through 45°.

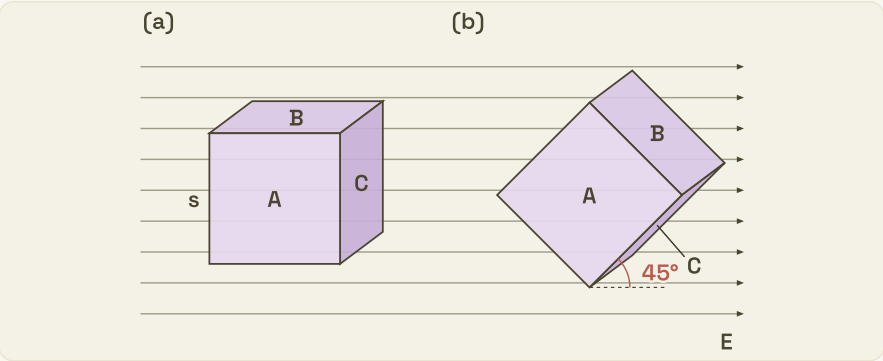


Figure 20.12 *A cube of side s in a uniform field E: (a) faces A and B lie along the field, C faces it; (b) the cube turned through 45° in the plane of the page.*

Hint

For each face, ask: what is the angle between its normal and the field? Each face has area s^2.

Answer

(a) Faces A (front) and B (top) lie along the field: their normals are at 90° to it, so $\Phi_A = 0$ and $\Phi_B = 0$. Face C faces the field squarely, its normal along it: $\Phi_C = s^2E$.

(b) Face A still lies along the field: $\Phi_A = 0$. Faces B and C are each at 45°: their normals make 45° with the field, so

$$\Phi_B = \Phi_C = s^2E\cos 45^\circ = \frac{s^2E}{\sqrt{2}}$$

Notice that in (b) the flux out through B and C together is $2s^2E/\sqrt{2} = \sqrt{2}\,s^2E$, more than through C in (a): the two slanting faces together are wider across the field. But for the whole closed cube, as much flux goes in through the left-hand faces as comes out through the right-hand ones: the total is zero, as 20.6 explains.

Check 4

A flat sheet of area 0.20 m² is in a uniform field of 500 N/C. Its normal makes 60° with the field. The flux through it is:

a 100 N m²/C
b 50 N m²/C
c 87 N m²/C
d 0

20.6 Gauss's law

Gauss's law comes from the inverse square nature of the force between point charges. It follows from Coulomb's law, yet it is often far more convenient: it gives the field of highly symmetrical arrangements of charge in a few lines, and it lets you reason about difficult problems without any calculation.

To see where it comes from, imagine a box made of a material that has no effect on electric fields.

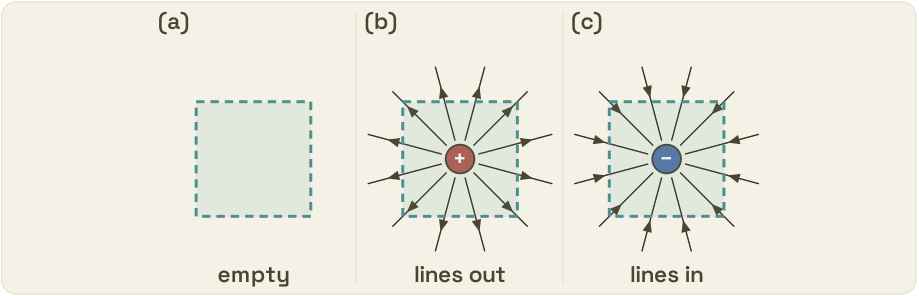


Figure 20.13 *Three imaginary boxes: (a) empty; (b) a positive charge inside; (c) a negative charge inside.*

- With the box **empty** (a), a positive test charge brought close to any face of it feels no force.
- With a **positive charge inside** (b), the test charge feels an **outward** force near every face. Put twice as much charge inside, and the outward force near every face doubles.
- With a **negative charge inside** (c), the test charge feels an **inward** force near every face; doubling the charge doubles it.

Thinking of the field lines as water, they seem to **flow out** of a box containing positive charge and **into** a box containing negative charge, in proportion to the charge.

Now put an **empty** box in a **uniform** field.

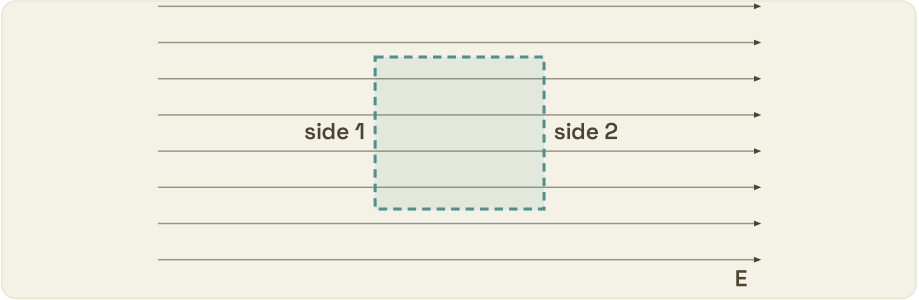


Figure 20.14 *An empty box in a uniform field: every line that enters through side 1 leaves through side 2.*

Near side 1 a positive test charge is pushed inwards; near side 2, outwards. The field is parallel to the other four sides, so near them there is no push in or out. In the water picture, as much flows in as flows out: the **net** flow is zero. Whenever there is charge in the box, there is a net flow of field lines out of it or into it; with no charge inside, the net flow is zero, however the field outside is shaped.

These observations, with flux as the measure of the "flow" of field lines, lead to **Gauss's law**:

$$\boxed{\Phi = \frac{q}{\varepsilon_0}}$$

where Φ is the total (net) electric flux out of a **closed** surface, and q is the **net charge inside** it. The closed surface, called a **Gaussian surface**, can be a box like ours or any closed shape at all. It is imaginary: you choose it, usually with the same symmetry as the charges, so that the field has the same size all over it.

In words: **the total electric flux through any closed surface equals the net charge inside the surface divided by ε_0.**

- Charges **outside** the surface add nothing to the total: each of their lines that enters the surface also leaves it.
- Only the **net** charge counts: +3 μC and −3 μC inside give zero total flux, though there is a field.
- The size and shape of the surface don't matter, only what it encloses.

Gauss's law is one of the **four fundamental equations** (Maxwell's equations) that govern electric and magnetic fields throughout the universe. Whether you travel into a star in a distant galaxy, down among the strands of a DNA molecule, or into the mi-