Unit 18

Dispersion

This unit online, with its interactives:
science-playground.com/s/11f18

Shine sunlight through a glass prism and a band of colors falls on the wall; look up after a shower with the Sun behind you and the same colors stand in the sky as a rainbow. White light is not one thing: it is a mixture, and glass and water bend each part of it by a slightly different amount. That small difference, a few percent, is **dispersion**.

This unit explains why it happens, how to measure it (angular dispersion and dispersive power), and the trouble it makes for lenses: a simple lens focuses violet light a little nearer than red, so its images are fringed with color. You will see how two lenses of different glasses, cemented together, cancel the colors, which is how every good camera, microscope and telescope lens is made.

A STORY FROM SCIENCE

The engineer who proved light is a wave

Augustin-Jean Fresnel (pronounced "fray-NEL") was born in Broglie, in Normandy (France), on 10 May 1788. He was slow to learn to read as a child, but he trained as a civil engineer and spent his working life building roads and bridges for the French government. He studied light in his spare time, and in 1815, when political trouble cost him his job for a while, he did his first experiments at his mother's home with a drop of honey as a lens.

In 1818 he entered a competition of the French Academy of Sciences on the diffraction of light, the way light spreads round edges, with a mathematical theory that treated light as a wave, as the English physicist Thomas Young had argued. One of the judges, the mathematician **Siméon-Denis Poisson**, believed Newton's idea that light is a stream of tiny particles. He worked out from Fresnel's equations what seemed an absurd result: in the shadow of a small round disc there should be a bright spot at the very center, nearly as bright as if the disc were not there. Another judge, **François Arago**, tried it, and the spot was there. Fresnel won the prize in 1819, and the spot is still called **Poisson's spot** (or Arago's spot). Fresnel went on to explain polarization and to invent the **Fresnel lens**, thin rings of glass that do the work of a huge, heavy lens, which made lighthouse beams visible tens of kilometers out to sea. He died of tuberculosis on 14 July 1827, aged 39.



Figure 18.1 *Augustin-Jean Fresnel (1788–1827), who showed that light behaves as a wave.*

Photo: E Rosette after a painting by A Tardieu, Public domain

18.1 Introduction

The colors we see round us, in a rainbow, in the sparkle of a diamond or a cut-glass bead, in the fringes round a cheap magnifying glass, come from **dispersion**. When white light meets a prism or a drop of water, the wavelengths it is made of are bent by slightly different amounts and spread into a band of colors from violet to red, each color a range of wavelengths. Isaac Newton first showed this carefully, in 1666, with prisms bought at a fair: one prism split sunlight into colors, and a second, upside down, put them back together into white. Dispersion explains rainbows and the colors of prisms, limits how sharp a lens can be, and, used on purpose, lets a spectrometer tell which elements a star is made of.

18.2 Dispersion of light

When a beam of white light falls on one face of a glass prism, it comes out split into a band of colors (Figure 18.2). Seven are usually named: **violet, indigo, blue, green, yellow, orange and red** (the order is easy to remember as VIBGYOR). The band of colors on the screen is called the **spectrum** of white light.

Dispersion of light is the splitting of white light into its constituent colors. The formation of a rainbow is an example.

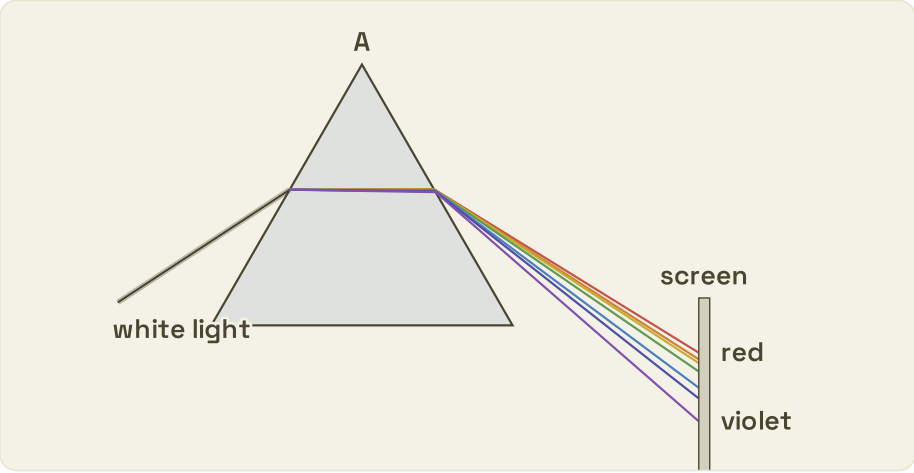


Figure 18.2 *White light dispersed by a glass prism. Each color is bent by a different angle: red least, violet most. (Each ray is traced with the real refractive indices of dense flint glass, from 1.774 for red to 1.839 for violet.)*

The seven names are a convention: the spectrum changes smoothly from one color to the next, with no edges. Newton chose seven, partly to match the seven notes of a musical scale; many scientists today leave out indigo.

18.3 Cause of dispersion

The refractive index of a medium depends on the wavelength (the color) of the light: it is larger for shorter wavelengths. For glass and water, and for most transparent materials, it follows closely **Cauchy's relation**, found by experiment:

$$\mu = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \dots$$

where A, B and C are constants for the material (B and C are small and positive). So μ is **not** inversely proportional to the wavelength, but it does **decrease** as the wavelength increases. Red light has the longest wavelength (about 700 nm) and violet the shortest (about 400 nm), so

$$\mu_v > \mu_i > \mu_b > \mu_g > \mu_y > \mu_o > \mu_r$$

In a small-angled prism of angle A the deviation is $\delta = (\mu - 1)A$ Grade 11, so a larger μ means a larger deviation: violet is deviated most and red least,

$$\delta_v > \delta_i > \delta_b > \delta_g > \delta_y > \delta_o > \delta_r$$

The colors that entered together therefore leave in different directions, and white light is split.

DID YOU KNOW?

How much does glass bend each color?

For the crown glass of ordinary lenses (Schott N-BK7), μ is 1.514 for red light of 680 nm, 1.517 for yellow and 1.530 for violet of 410 nm, a difference of only 1%. Dense flint glass, which has lead or titanium compounds in it, both bends more and spreads more: 1.774 for red, 1.839 for violet. That is why prisms for spectroscopes are made of flint glass.

Why does the index depend on wavelength at all? Light slows down in glass because its electric field makes the electrons in the glass's atoms vibrate, and their vibrations send out waves of their own that combine with the light. The electrons have natural frequencies in the ultraviolet. Violet light is nearer those frequencies than red, so it drives the electrons harder and is slowed more: its refractive index is larger.

Check 1

In a glass prism, which light has the larger speed?

a Red, because glass has a smaller refractive index for red

b Violet, because glass has a larger refractive index for violet

c Both have the same speed in glass, since they have the same speed in air

d Red, because red has the higher frequency

Results of dispersion

a. Rainbow

A rainbow is a spectrum made by raindrops. It appears when the Sun is behind you and fairly low and there are raindrops in front of you. A ray of sunlight entering a drop is refracted, and dispersed, at its surface, is reflected once off the back of the drop, and is refracted again as it comes out (Figure 18.3). The reflection is an ordinary partial reflection, not total internal reflection: most of the light goes straight out of the back of the drop, and only a few percent is reflected.

Rays entering the drop at different heights leave at different angles, but they crowd together near a largest angle, about 42° for red and 40° for violet, so most of the light comes out there. You see red light from all the drops that lie 42° from the point directly opposite the Sun, the shadow of your head: those drops make a red circle. Violet comes from drops at 40°, a slightly smaller circle inside. So a rainbow is always a circle round the shadow of your head, red outside and violet inside; the ground usually hides its lower part. From an airplane you can sometimes see the whole circle.

A fainter **secondary rainbow** at about 51° comes from light reflected twice inside each drop; its colors are the other way round, red inside. The sky between the two bows is darker, because no light reaches you from drops there after one or two reflections.

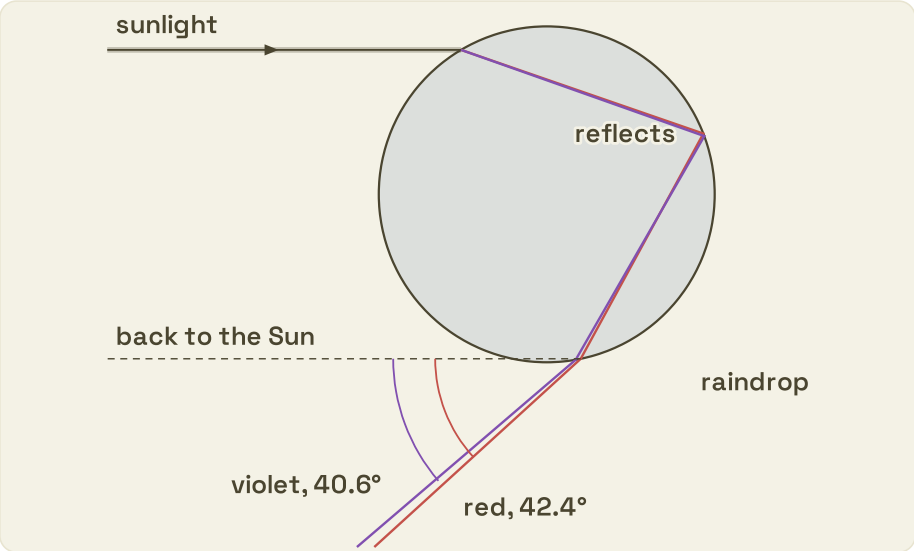


Figure 18.3 *Light in a raindrop. Sunlight enters near the top, reflects off the back and leaves split: red at 42.4° to the line back to the Sun, violet at 40.6° (the bow's two edges, usually rounded to 42° and 40°). The ray is traced with water's real indices; this height is the one that sends the most light out at these angles.*

b. The blue color of the sky

Sunlight passing through the air is **scattered** by the molecules of nitrogen and oxygen: they take a little of the light and send it out in all directions. Lord Rayleigh showed that for particles much smaller than the wavelength, the intensity of the scattered light varies as

$$I \propto \frac{1}{\lambda^4}$$

Blue light, of wavelength about 450 nm, is scattered about $(700/450)^4 \approx 6$ times as strongly as red light of 700 nm. Whichever way you look, away from the Sun, you see sunlight scattered towards you by the air, and most of it is blue: the sky looks blue. (It isn't violet, though violet is scattered even more, because sunlight has less violet than blue, and our eyes are much less sensitive to it.) Above the atmosphere, on the Moon, there is nothing to scatter the light, and the sky is black even in daylight.



Figure 18.4 *A double rainbow. The primary bow has red on the outside; in the fainter secondary bow the colors are reversed, and the sky between the two is darker.*
Photo: Eric Rolph at English Wikipedia, CC BY-SA 2.5

c. The red Sun at sunrise and sunset

When the Sun is near the horizon its light passes through much more air to reach us, about 38 times as much as when it is overhead. Along the way most of the blue and green is scattered out of the beam, and what comes straight through is mainly orange and red. So the rising and setting Sun, and the clouds it lights, look red. Dust and smoke in the air make the sunsets redder still; in the hazy air of the Tarai and of Kathmandu in winter, the Sun often turns red well before it reaches the horizon.

14. a. Explain how two prisms can produce dispersion without deviation, and deviation without dispersion.
    b. On what factors do angular dispersion and dispersive power depend?
    c. When white light is dispersed by a prism, red appears at the top of the spectrum and violet at the bottom. Why?
    d. The dispersive power of crown glass is 0.013 and that of flint glass 0.026. Calculate the focal lengths of the lenses that form an achromatic doublet of focal length 100 cm when placed in contact.

15. a. How would you combine two lenses of different materials so that there is no chromatic aberration?
    b. An achromatic converging lens of focal length 30 cm is to be made by combining a crown glass lens and a flint glass lens. What must the focal lengths of the two lenses be, if the dispersive power of crown glass is 0.018 and that of flint glass 0.027?

Numerical problems

16. A concave lens made of crown glass has radii of curvature 25 cm and 30 cm. Compute its longitudinal chromatic aberration, given $\mu_v = 1.523$ and $\mu_r = 1.5204$.

17. An achromatic combination of a convex lens of crown glass and a concave lens of flint glass has a power of +5 D. The dispersive power of crown glass is 0.015 and that of flint glass 0.030. Calculate the focal lengths of the two lenses.

18. The dispersive power of crown glass is 0.015 and that of flint glass 0.030. Calculate the focal lengths of the crown and flint lenses that form an achromatic doublet of focal length 80 cm when placed in contact.

19. An achromatic combination of focal length 50 cm consists of a convex lens of crown glass and a concave lens of flint glass. Find the focal length of each lens from these data: crown glass $\mu_v = 1.523$, $\mu_r = 1.517$; flint glass $\mu_v = 1.664$, $\mu_r = 1.650$.

20. Calculate the angular dispersion between violet and red produced by a small-angled prism of angle A = 5°, with $\mu_v = 1.66$ and $\mu_r = 1.54$.

21. The refractive indices of crown glass for violet and red light are 1.523 and 1.513, and those of flint glass 1.773 and 1.743. Calculate their dispersive powers.

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

Check 11

8. The dispersive powers of crown and flint glass are 0.015 and 0.03. What are the focal lengths of the lenses that form an achromatic doublet of focal length 100 cm when placed in contact?

a 50 cm and −100 cm
b 90 cm and −180 cm
c 100 cm and −200 cm
d 200 cm and −400 cm

12.

a. How is dispersive power related to the refractive index of a material?
b. Why does red light appear at the top of the spectrum and violet at the bottom when white light is dispersed by a prism?
c. The dispersive powers of crown and flint glass are 0.015 and 0.03. Calculate the focal lengths of the lenses that form an achromatic doublet of focal length 100 cm when placed in contact.

13.

a. What do you mean by chromatic aberration?
b. Explain the meaning of achromatism in a lens.
c. Find the angle of a prism of dispersive power 0.03 and refractive index 1.53 that forms an achromatic combination with another prism of angle 5°, dispersive power 0.045 and refractive index 1.66.

CAREFUL

The sky's blue is scattering, not dispersion

Some books say the sky is blue, or the Sun red at sunset, "due to dispersion". It isn't: no refraction splits the light into a spectrum there. The air **scatters** short wavelengths more than long ones (Rayleigh scattering), and that alone explains both. Dispersion is the dependence of the refractive index on wavelength; scattering is light being sent off in new directions by particles. Write "scattering" in an exam.

18.4 Spectrum of light

A **spectrum** is the band of colors obtained when light is split into its wavelengths. When white light is dispersed by a prism, each color leaves at its own angle; caught on a screen or a camera's sensor, the colors form a spectrum, which shows which wavelengths the light contains. There are two kinds: pure and impure.

Pure spectrum

A **pure spectrum** is one in which the colors do not overlap: each color falls at its own place, and all can be seen distinctly. To get one, the light must be sent through the prism as a **parallel beam**, and each color then focused to its own point, as in a **spectrometer** (Figure 18.5). A narrow slit, lit from behind, is placed at the principal focus of a converging lens (the collimator), so a parallel beam leaves the lens and falls on the prism. The prism splits it into separate parallel beams of different colors, each going its own way, and a second lens brings each to its own focus in its focal plane. The pure spectrum forms there, on a white screen, or is looked at through a magnifying eyepiece.

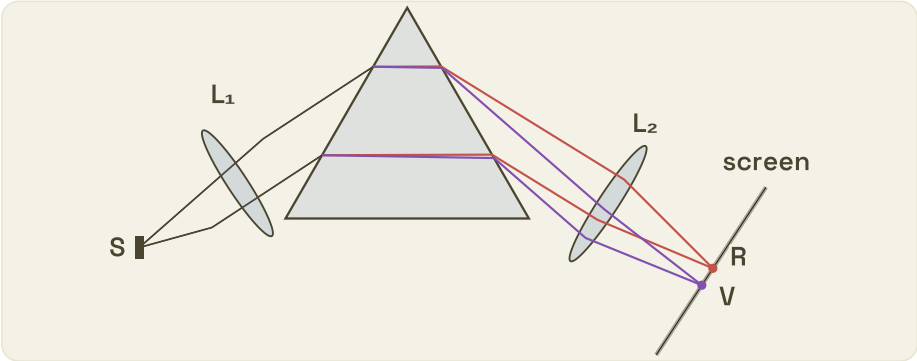


Figure 18.5 *A pure spectrum. The slit S is at the focus of lens L_1, so a parallel beam meets the prism; each color leaves as its own parallel beam, and lens L_2 brings red to R and violet to V, apart. (Only red and violet are drawn, traced through dense flint glass.)*

So the conditions for a pure spectrum are:

1. The slit must be narrow (a wide slit gives each color a wide image, and neighboring colors overlap).
2. The light falling on the prism must be a parallel beam (the slit at the focus of the first lens).
3. The prism must be set at minimum deviation (then the parallel beam stays parallel and the images are sharpest).
4. The emerging beams must be focused by an achromatic lens (section 18.8), so that the lens itself doesn't separate the colors further.

Impure spectrum

An **impure spectrum** is one in which the colors overlap, so they can't be seen distinctly. It is what you get when light spreading from a source, or from a wide slit, falls straight on a prism: each color leaves as a spreading beam, and the patches of different colors fall on top of one another (Figure 18.6). The middle of such a spectrum is nearly white, with colored edges.

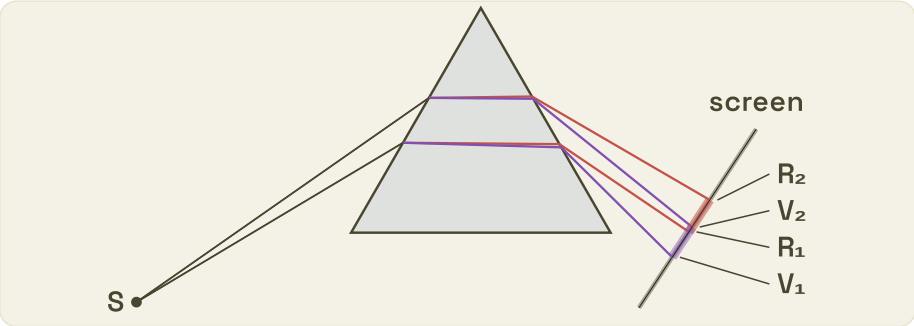


Figure 18.6 *An impure spectrum. Light spreading from S meets the prism; on the screen, red covers R_1R_2 and violet V_1V_2, and the two overlap.*

18.5 Angular dispersion

Angular dispersion is the difference in the angles of deviation of the two extreme colors of the spectrum, violet and red (Figure 18.7):

$$\theta = \delta_v - \delta_r$$

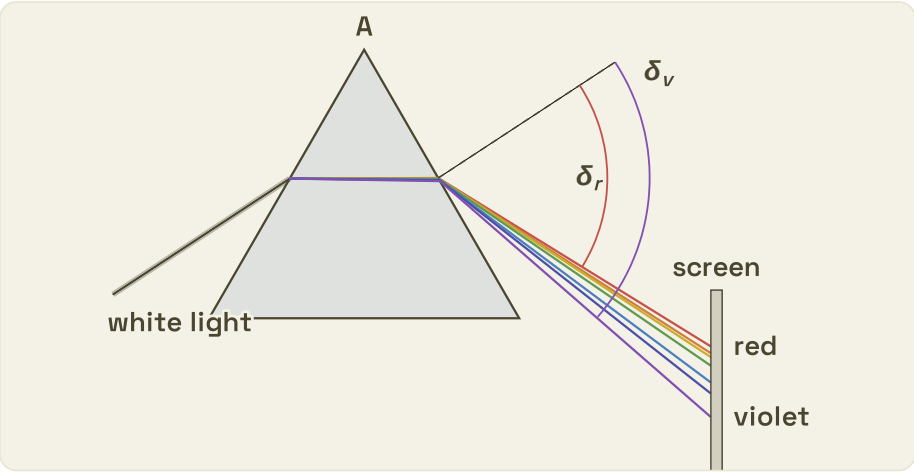


Figure 18.7 *The deviations of red and violet, measured from the direction the white light came in (dashed). The angle between the two emerging rays is the angular dispersion, $\delta_v - \delta_r$.*

Yellow light is deviated by about the average of the deviations of all the colors, so it is taken as the **mean color**, and the mean deviation is that of yellow: $\delta_r < \delta_y < \delta_v$.

For a small-angled prism of angle A, $\delta_v = (\mu_v - 1)A$ and $\delta_r = (\mu_r - 1)A$, so

Check 7

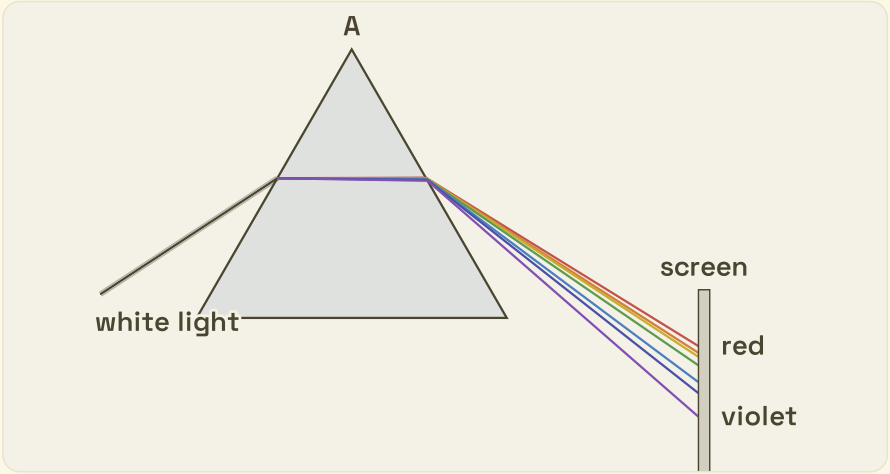


Figure 18.12 *For question 4.*

4. What does the figure above show?

a refraction of light
b dispersion of white light into seven colors
c deviation produced for white light
d chromatic aberration of light by a prism

Check 8

5. A prism of dispersive power 0.03 and refractive index 1.53 forms an achromatic combination with another prism of angle 5°, dispersive power 0.045 and refractive index 1.66. What is the angle of the first prism?

a 8.34°
b 9.34°
c 10.23°
d 11.54°

Check 9

6. For a convex lens, $f_v = 25$ cm and $f_r = 26$ cm (Figure 18.8 shows the arrangement). What is the dispersive power of its glass?

a -0.39
b 0.039
c 0.049
d 0.39

Check 10

7. An achromatic converging lens of mean focal length 40 cm is made by combining two lenses of different materials whose dispersive powers are in the ratio 1 : 3. What is the focal length of each lens?

a 24 cm, −80 cm
b 26 cm, −80 cm
c $26\frac{2}{3}$ cm, −80 cm
d $26\frac{2}{3}$ cm, −90 cm

IN SHORT

Key ideas of Unit 18

- **Dispersion** is the splitting of white light into its colors. It happens because the refractive index decreases with wavelength (Cauchy: $\mu = A + B/\lambda^2 + \ldots$): violet is bent most, red least.
- A **rainbow** is sunlight refracted, reflected once and refracted again in raindrops: red at about 42° from the point opposite the Sun, violet at 40°. The blue sky and the red setting Sun come from **scattering** ($I \propto 1/\lambda^4$), not dispersion.
- A **pure spectrum** needs a narrow slit, a parallel beam, the prism at minimum deviation and an achromatic lens; otherwise the colors overlap (impure).
- **Angular dispersion** $\theta = \delta_v - \delta_r = (\mu_v - \mu_r)A$; **dispersive power** $\omega = \dfrac{\delta_v - \delta_r}{\delta} = \dfrac{\mu_v - \mu_r}{\mu - 1}$, a property of the material alone.
- **Chromatic aberration** of a lens: $f_r - f_v = \omega f$. **Spherical aberration:** marginal rays focus nearer than paraxial ones; reduced by stops, bending, lens combinations and aspheric surfaces.
- **Achromatic doublet:** $\dfrac{\omega}{f} + \dfrac{\omega'}{f'} = 0$: a convex crown lens with a concave flint lens. Two prisms can give deviation without dispersion, $\omega(\mu - 1)A = \omega'(\mu' - 1)A'$, or dispersion without deviation, $(\mu - 1)A = (\mu' - 1)A'$.

Exercise

Choose the correct option

Check 4

1. Which of the following characterizes the color of light?

a frequency
b wavelength
c velocity
d amplitude

Check 5

2. A lens with chromatic aberration forms the image of a white object. Which of the following is true about the image?

a colored
b black
c white
d yellow

Check 6

3. Which of the following must be true for a combination of lenses that removes chromatic aberration?

a $f_r + f_v = 0$
b $f_r > f_v$
c $f_r < f_v$
d $f_r - f_v = 0$

Answer these questions

9.

a. What is the cause of dispersion?

b. Why is there no dispersion of monochromatic light?

c. Explain why a mirror cannot give rise to chromatic aberration.

$$\theta = \delta_v - \delta_r = (\mu_v - 1)A - (\mu_r - 1)A = (\mu_v - \mu_r)A$$

The angular dispersion depends on the material (through $\mu_v - \mu_r$) and on the angle of the prism.

WORKED EXAMPLE

Angular dispersion of a crown glass prism

A small-angled prism of crown glass has an angle of 8°. Its refractive indices are 1.530 for violet and 1.514 for red. Find the deviations of the two colors and the angular dispersion.

Answer

$\delta_v = (1.530 - 1) \times 8^\circ = 4.24^\circ$, $\delta_r = (1.514 - 1) \times 8^\circ = 4.11^\circ$.
$\theta = \delta_v - \delta_r = (1.530 - 1.514) \times 8^\circ =$ **0.13°**: the colors are spread over only an eighth of a degree, while the light as a whole is turned by about 4°.

18.6 Dispersive power

White light passing through a prism is both **deviated** and **dispersed**. The **dispersive power** of a material, ω, compares the two: it is the ratio of the angular dispersion between the extreme colors to the deviation of the mean color.

$$\omega = \frac{\delta_v - \delta_r}{\delta}$$

For a small-angled prism, $\delta_v - \delta_r = (\mu_v - \mu_r)A$ and $\delta = (\mu - 1)A$, where μ is the index for the mean (yellow) color, so A cancels:

$$\omega = \frac{\mu_v - \mu_r}{\mu - 1}$$

Dispersive power is a ratio of two angles, so it has no unit and no dimensions. Unlike the angular dispersion, it doesn't depend on the prism's angle: it is a property of the material alone (for the two colors chosen).

DID YOU KNOW?

Crown and flint

Optical glass makers quote dispersive power between two particular colors, the blue F line (486 nm) and the red C line (656 nm) of hydrogen, with yellow at 589 nm as the mean. On that scale, crown glass (N-BK7) has ω ≈ 0.016 and dense flint (N-SF11) ω ≈ 0.039: flint spreads colors more than twice as much for the same bending. Taken between the very ends of the spectrum, violet 410 nm and red 680 nm, the same glasses give about 0.031 and 0.083. So a value of ω means something only with the colors it was measured for; the exercises of this unit give ω directly.
Diamond has a high refractive index (2.42) but a dispersive power of only about 0.018 between the C and F lines, close to crown glass. Its "fire", the flashes of color, comes from its high index: the deviations are large, so even a modest fraction of them is a wide spread of colors, and the cut sends light back and forth inside the stone, through face after face, before it leaves.

Check 2

Two small-angled prisms are cut from the same glass, one of angle 4° and one of 8°. Compared with the 4° prism, the 8° prism has:

a twice the angular dispersion and twice the dispersive power

b twice the angular dispersion and the same dispersive power

c the same angular dispersion and twice the dispersive power

d the same angular dispersion and the same dispersive power

18.7 Aberration of light in lenses

The lens formula assumes a perfect lens: every ray from a point of the object meets again at one point of the image. A real lens falls short of that. **Aberration** is the difference in the shape, size or position of the actual image from the one the lens formula predicts. Because of it, light from one point is spread over a small region instead of being brought to a point, and the image is blurred. The two main kinds are **chromatic aberration** and **spherical aberration**.

a. Chromatic aberration

A lens can be thought of as a stack of prisms, so it disperses light like a prism. Rays of white light parallel to the axis are split as they are refracted, and each color crosses the axis at its own point (Figure 18.8). **Chromatic aberration** is the inability of a lens to bring all colors of light to a focus at a single point on the principal axis. This kind is also called **axial** or **longitudinal** chromatic aberration, and it is measured by the difference between the focal lengths for red and violet light:

$$\text{chromatic aberration} = f_r - f_v$$

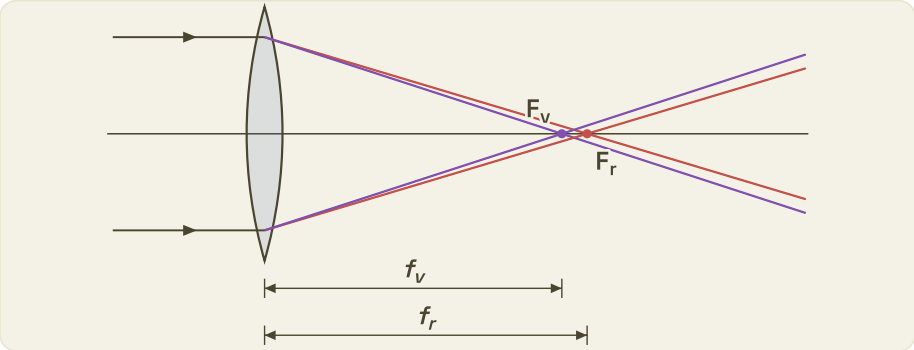


Figure 18.8 *Chromatic aberration in a convex lens. Violet is bent more and focuses at F_v, nearer the lens than red's F_r. (A single lens of dense flint glass, with the real spread between red and violet.)*

So a white object seen through a simple lens has colored edges: focus the screen for red and a violet blur surrounds the image, focus for violet and the fringe is red.

Deriving the chromatic aberration

Let f_r, f and f_v be the focal lengths, and μ_r, μ and μ_v the refractive indices, for red, mean (yellow) and violet light, for a thin lens whose faces have radii of curvature R_1 and R_2.

For the mean color. By the lens maker's formula,

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad \text{so} \quad \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{f(\mu - 1)} \qquad (1)$$

WORKED EXAMPLE

A crown and flint doublet

An achromatic lens of focal length +30 cm is to be made by combining a crown glass lens and a flint glass lens. What must the focal lengths of the two lenses be, if the dispersive powers of crown and flint glass are 0.018 and 0.027?

Answer

$F = 30$ cm, $\omega_1 = 0.018$ (crown), $\omega_2 = 0.027$ (flint).

Achromatic condition: $\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0$, so $\frac{f_1}{f_2} = -\frac{\omega_1}{\omega_2} = -\frac{0.018}{0.027} = -\frac{2}{3}$, and $f_1 = -\frac{2}{3} f_2$ … (1)

Combination: $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} = -\frac{3}{2f_2} + \frac{1}{f_2} = -\frac{1}{2f_2}$.

So $\frac{1}{30} = -\frac{1}{2f_2}$, giving $f_2 = \mathbf{-15\ cm}$ (the flint lens, concave), and from (1), $f_1 = -\frac{2}{3}(-15) = \mathbf{+10\ cm}$ (the crown lens, convex).

Dispersion without deviation (a direct-vision prism). The deviations of the mean color cancel if

$$(\mu - 1)A = (\mu' - 1)A'$$

Yellow light then goes straight through, while the other colors are still spread, by the net dispersion $(\mu_v - \mu_r)A - (\mu'_v - \mu'_r)A'$. The pocket spectroscope in many school laboratories uses a direct-vision prism, usually of three or five prisms cemented together, so it can be pointed straight at the lamp.

WORKED EXAMPLE

An achromatic pair of prisms

A prism of crown glass, of dispersive power 0.03 and refractive index 1.53, is to be combined with a prism of angle 5°, dispersive power 0.045 and refractive index 1.66, so that the pair produces no dispersion. Find the angle of the crown prism.

Answer

For no dispersion, $\omega(\mu - 1)A = \omega'(\mu' - 1)A'$:

$0.03 \times 0.53 \times A = 0.045 \times 0.66 \times 5^\circ$

$A = \frac{0.1485}{0.0159} = \mathbf{9.34°}$.

The net deviation is $0.53 \times 9.34^\circ - 0.66 \times 5^\circ = 4.95^\circ - 3.30^\circ = 1.65^\circ$: the light is turned, but not colored.

Check 3

An achromatic doublet of focal length +40 cm is made from two glasses whose dispersive powers are 0.02 and 0.04. Which pair of lenses works?

a a convex lens of +20 cm ($\omega = 0.02$) and a concave lens of −40 cm ($\omega = 0.04$)

b a convex lens of +20 cm ($\omega = 0.04$) and a concave lens of −40 cm ($\omega = 0.02$)

c two convex lenses of +80 cm, one of each glass

d a convex lens of +40 cm ($\omega = 0.02$) and a concave lens of −40 cm ($\omega = 0.04$)

$$\frac{\mu_v - \mu_r}{f(\mu - 1)} + \frac{\mu'_v - \mu'_r}{f'(\mu' - 1)} = 0$$

With $\omega = \dfrac{\mu_v - \mu_r}{\mu - 1}$ and $\omega' = \dfrac{\mu'_v - \mu'_r}{\mu' - 1}$:

$$\frac{\omega}{f} + \frac{\omega'}{f'} = 0$$

This is the condition for an achromatic combination of two thin lenses in contact. It tells us three things:

1. Since ω and ω′ are both positive, f and f′ must have **opposite signs**: one lens convex and the other concave.
2. The lenses must be of **different materials**. If $\omega = \omega'$, the condition gives $f = -f'$, and the combination's power $1/f + 1/f'$ would be zero: no lens at all.
3. $\dfrac{f'}{f} = -\dfrac{\omega'}{\omega}$: the lens of larger dispersive power has the longer focal length. For a converging doublet the convex lens must be the stronger one, so it is made of the glass of **smaller** dispersive power (crown), and the concave lens of the glass of larger dispersive power (flint).

DID YOU KNOW?

Who made the first achromatic lens?

Newton thought chromatic aberration could not be cured, since he believed every glass spread colors in the same proportion to its bending, and he built a telescope with a mirror instead (a mirror reflects all colors alike). An English lawyer, **Chester Moor Hall**, found around 1730 that crown and flint glass differ, and had achromatic lenses made, but kept them secret. The London optician **John Dollond** developed the idea independently, patented the achromatic lens in 1758, and his firm made them for a century. Telescopes and microscopes became sharp and free of color; a doublet is still the objective of most small telescopes and binoculars.

Applications of achromatism

Achromatic lenses bring different colors of light to one focus, which gives sharp images with true colors. They are used wherever images matter:

1. **Cameras:** every camera lens, including a phone's, is a group of lenses of different glasses (or plastics) designed so that the colors focus together; otherwise photos would have colored fringes round every edge.
2. **Optical instruments:** binoculars, spotting scopes, telescopes, microscopes, rangefinders and surveying instruments use achromatic objectives for sharp, color-correct images; a spectrometer uses them to focus each color of its spectrum accurately.
3. **Optical fiber systems:** achromatic lenses couple light into and out of fibers, so light of several wavelengths carried by one fiber is focused onto it equally well.
4. **Projectors:** overhead, film and digital projectors use achromatic lenses so the picture on the screen is sharp and its colors stay together.

Two prisms: deviation without dispersion, and dispersion without deviation

Prisms can be combined in the same way. Put a prism of angle A (indices μ, μ_v, μ_r) and a small prism of another glass, angle A′, upside down against it, so that their deviations and dispersions are in opposite directions.

Deviation without dispersion (an achromatic prism). The dispersions cancel if

$$(\mu_v - \mu_r)A = (\mu'_v - \mu'_r)A', \quad \text{or, with } \omega = \frac{\mu_v - \mu_r}{\mu - 1}: \quad \omega(\mu - 1)A = \omega'(\mu' - 1)A'$$

White light then comes out white, but turned by the net deviation $(\mu - 1)A - (\mu' - 1)A'$, which isn't zero because the glasses differ.

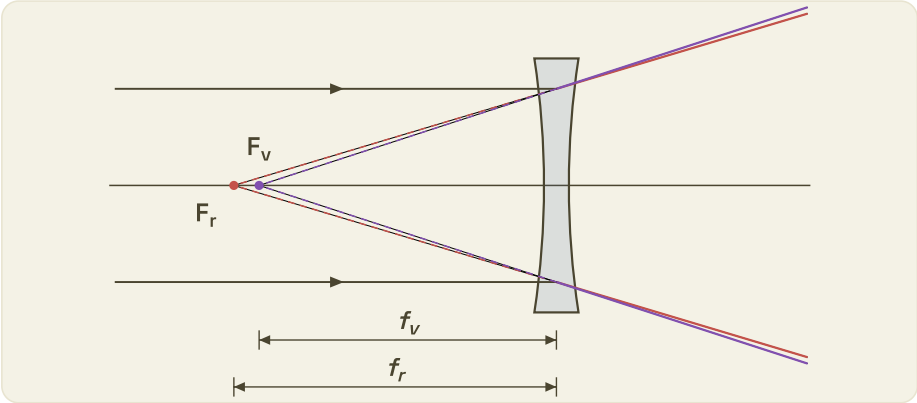


Figure 18.9 *Chromatic aberration in a concave lens. The rays spread out; traced back, violet seems to come from F_v, nearer the lens than red's F_r.*

For red light. The radii are the same, so using (1),

$$\frac{1}{f_r} = (\mu_r - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{\mu_r - 1}{f(\mu - 1)} \qquad (2)$$

For violet light, in the same way,

$$\frac{1}{f_v} = \frac{\mu_v - 1}{f(\mu - 1)} \qquad (3)$$

Subtracting (2) from (3):

$$\frac{1}{f_v} - \frac{1}{f_r} = \frac{(\mu_v - 1) - (\mu_r - 1)}{f(\mu - 1)}, \quad \text{so} \quad \frac{f_r - f_v}{f_r f_v} = \frac{\mu_v - \mu_r}{f(\mu - 1)}$$

Using the dispersive power. $\omega = \dfrac{\mu_v - \mu_r}{\mu - 1}$, so

$$f_r - f_v = \frac{\omega}{f} f_r f_v$$

The focal lengths differ by only a few percent, so the product $f_r f_v$ is very nearly f^2 (the mean focal length is almost exactly the geometric mean of the two, $f \approx \sqrt{f_r f_v}$). Then

$$f_r - f_v = \omega f$$

The chromatic aberration of a thin lens is the product of its dispersive power and its mean focal length. A lens of short focal length has a small chromatic aberration in centimeters, but no single lens is free of it. For a concave lens f is negative, so $f_r - f_v$ is negative too: its violet focus is again nearer the lens, as in Figure 18.9.

b. Spherical aberration

Spherical aberration is the inability of a lens to bring **paraxial rays** (rays close to the axis) and **marginal rays** (rays through the zone near the lens's rim) to a focus at the same point. It happens even with light of one color, because the lens's faces are parts of spheres: near the rim, a spherical surface meets the rays at angles too steep for them all to be bent to the paraxial focus. In a convex lens the marginal rays are bent too much and cross the axis **nearer** the lens than the paraxial rays (Figure 18.10).

The same happens for a point object O on the axis: its paraxial rays meet at a point I_p, farther from the lens, and its marginal rays at I_m, nearer. The image isn't sharp anywhere along the axis. A screen placed across the axis shows a disc of light instead of a point; the smallest such disc, between I_m and I_p, is the **circle of least**

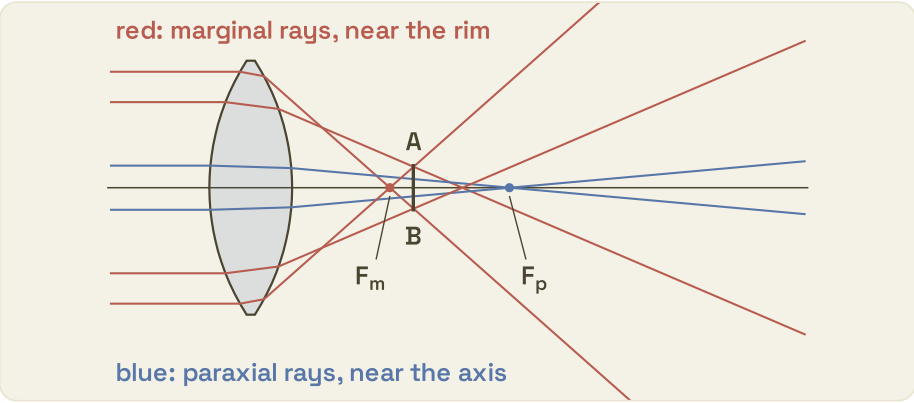


Figure 18.10 *Spherical aberration in a convex lens. Marginal rays (red) cross the axis at F_m, nearer the lens than the paraxial rays (blue) at F_p. The beam is narrowest at AB, the circle of least confusion. (Rays traced exactly through a lens of glass $\mu = 1.5$ with spherical faces.)*

confusion (AB in the figure), where the image is best. Lenses with a large aperture (a wide opening) are affected most; their images get sharper as the aperture is made smaller.

In a concave lens the marginal rays are also bent more, so traced back they seem to come from a point nearer the lens than the paraxial focus (Figure 18.11). By custom the aberration is called positive for a convex lens and negative for a concave one.

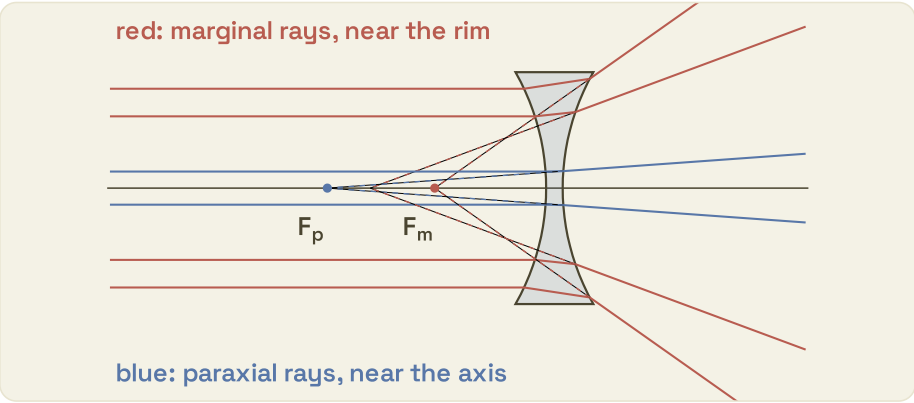


Figure 18.11 *Spherical aberration in a concave lens. Traced back, the marginal rays (red) seem to come from F_m, nearer the lens than F_p.*

To remove spherical aberration completely, every zone of the lens would need the same focal length, and with spherical faces that is impossible: a single spherical lens can only have its spherical aberration **reduced**. Ways of doing it:

1. **A stop:** cover the outer zone of the lens with a ring (an aperture stop), so only paraxial rays pass. The image is sharper but dimmer; this is why a camera's photos get sharper when its aperture is made smaller.
2. **Sharing the bending:** a lens bends rays least wastefully when they are bent about equally at its two faces. A plano-convex lens with its curved face towards the parallel light has much less aberration than the same lens turned round; this is why a magnifying glass or a lens for the Sun's rays should face its curved side to the light.

3. **Two lenses:** a convex and a concave lens can be combined so that their aberrations, of opposite sign, cancel (the same pair is usually made achromatic at the same time), or two plano-convex lenses separated by a distance equal to the difference of their focal lengths can share the bending.
4. **Aspheric surfaces:** a face that is not part of a sphere, but flattens towards the rim, can bring every zone to one focus. Grinding glass to such a shape was once very hard; today aspheric lenses are molded in plastic or glass by the million, and nearly every phone camera's lenses are aspheric.

WORKED EXAMPLE

Dispersive power from two focal lengths

For red light the focal length of a convex lens is found to be 26 cm; with violet light it becomes 25 cm. What is the dispersive power of the lens's glass, and what is the chromatic aberration?

Answer

$f_r = 26$ cm, $f_v = 25$ cm. Take the mean focal length as $f = \dfrac{f_r + f_v}{2} = \dfrac{26 + 25}{2} = 25.5$ cm (the geometric mean, $\sqrt{26 \times 25} = 25.50$ cm, is the same to this accuracy).
Chromatic aberration $= f_r - f_v = 26 - 25 = \mathbf{1\ cm}$.
$f_r - f_v = \omega f$, so $\omega = \dfrac{f_r - f_v}{f} = \dfrac{1}{25.5} = \mathbf{0.039}$.

18.8 Achromatic combination of lenses

All the colors of white light can be brought to a focus at a single point on the principal axis by a suitable combination of lenses. A combination of two thin lenses that is free from chromatic aberration is called an **achromatic combination** of lenses, or an **achromatic doublet**; the property is called **achromatism**. The usual doublet is a convex lens of crown glass cemented to a weaker concave lens of flint glass. The flint lens spreads the colors as much as the crown lens does, the other way, while bending the light as a whole less, so the pair still converges, with the colors brought together.

Deriving the condition

Let two thin lenses L and L′, in contact, have dispersive powers ω and ω′. For lens L let μ_v, μ, μ_r be the refractive indices and f_v, f, f_r the focal lengths for violet, mean and red light; for L′ let the same quantities be primed.

For each lens, red light. As in the derivation of chromatic aberration (equation (2) there),

$$\frac{1}{f_r} = \frac{\mu_r - 1}{f(\mu - 1)}, \qquad \frac{1}{f'_r} = \frac{\mu'_r - 1}{f'(\mu' - 1)}$$

The combination for red light. Its focal length F_r is given by $\dfrac{1}{F_r} = \dfrac{1}{f_r} + \dfrac{1}{f'_r}$, so

$$\frac{1}{F_r} = \frac{\mu_r - 1}{f(\mu - 1)} + \frac{\mu'_r - 1}{f'(\mu' - 1)} \qquad (1)$$

For violet light, in the same way,

$$\frac{1}{F_v} = \frac{\mu_v - 1}{f(\mu - 1)} + \frac{\mu'_v - 1}{f'(\mu' - 1)} \qquad (2)$$

The condition. The combination is achromatic if red and violet are focused at the same point, $F_r = F_v$, so (1) and (2) are equal:

$$\frac{\mu_v - 1}{f(\mu - 1)} - \frac{\mu_r - 1}{f(\mu - 1)} + \frac{\mu'_v - 1}{f'(\mu' - 1)} - \frac{\mu'_r - 1}{f'(\mu' - 1)} = 0$$