Unit 17

Lenses

This unit online, with its interactives:
science-playground.com/s/11f17

Hold a pair of reading glasses over a sheet of paper in the sun and you can burn a hole in it; hold a short-sighted friend's glasses the same way and nothing happens. Both are curved pieces of glass. What decides whether a lens gathers light or spreads it, how strongly, and where its image will be?

This unit answers that with a few formulas. You will meet the lens formula, $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$ (the same as the mirror's), and the **lens maker's formula**, which says how a lens's focal length comes from its glass and the curves of its faces, and so why a lens in water is weaker and an air bubble in water spreads light. You will combine lenses, measure their power in diopters, and see how a magnifying glass makes a thing look bigger: by making it fill a larger angle at your eye.

A STORY FROM SCIENCE

The glassmaker who read the Sun

Joseph von Fraunhofer was born in Straubing, in Bavaria (Germany), on 6 March 1787, the eleventh child of a glazier. Orphaned at eleven, he was apprenticed to a mirror-maker and glass-cutter in Munich, and in 1801 the workshop collapsed on him; he was pulled alive from the rubble, and the Prince-Elector of Bavaria, who had watched the rescue, gave him money to buy books and a glass-grinding machine.
He went on to make the best optical glass and the finest refracting telescopes in Europe. To test his glass he needed light of a single, exact color, and while looking at sunlight through a prism and a small telescope he found the Sun's spectrum crossed by hundreds of dark lines. He mapped 574 of them, labeled the strongest A to K (letters still used), built some of the first diffraction gratings and used them to measure the lines' wavelengths, and used them to measure the refractive index of his glasses color by color. Those dark lines turned out to be the fingerprints of the elements in the Sun's atmosphere. Fraunhofer died of tuberculosis in Munich on 7 June 1826, only 39 years old.



Figure 17.1 *Joseph von Fraunhofer (1787–1826), optician, maker of the finest lenses of his day and discoverer of the dark lines in the Sun's spectrum.*
Photo: Wikimedia Commons, Public domain

17.1 Introduction

A **lens** is a piece of a uniform (homogeneous) transparent material, such as glass or plastic, bounded by two refracting surfaces, at least one of which is curved. Most lenses have spherical surfaces: each face is a part of a sphere (one face may be flat, which is a sphere of infinite radius).

Lenses are everywhere: eyeglasses and contact lenses, the cameras in phones, microscopes, binoculars and telescopes, projectors, reading and magnifying glasses, the headlights, spotlights and torches of vehicles, and the eyepieces of binoculars and microscopes. Your own eye has two: the **cornea** at its front, which does most of the bending, and the **crystalline lens** behind the pupil, which changes shape to focus. There are two main kinds of lens: convex and concave.

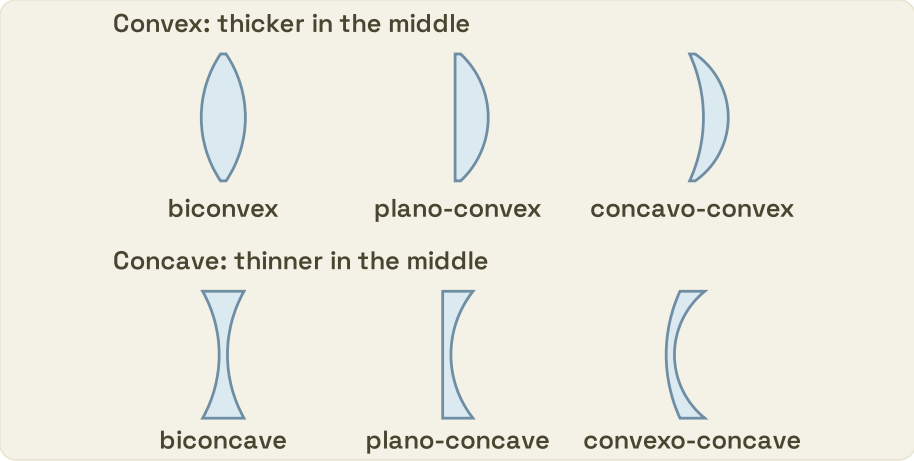


Figure 17.2 *The six kinds of lens, cut through the middle. Top: convex lenses, thicker in the middle (biconvex, plano-convex, concavo-convex). Bottom: concave lenses, thinner in the middle (biconcave, plano-concave, convexo-concave).*

1. Convex lens

A lens that is **thicker in the middle than at its edges** is a **convex lens**. There are three kinds (Figure 17.2, top row):

- **Biconvex lens**: both surfaces convex (bulging outwards). Their radii of curvature may or may not be equal; if they are, it is an **equiconvex** lens.
- **Plano-convex lens**: one surface flat (plane), the other convex.
- **Concavo-convex lens**: one surface concave, the other convex, with the **convex surface curved more** (smaller radius of curvature) than the concave one, so the lens is still thicker in the middle. A lens of this shape (convex on one side, concave on the other) is also called a **meniscus**; most spectacle lenses are menisci.

2. Concave lens

A lens that is **thinner in the middle than at its edges** is a **concave lens**. Again there are three kinds (Figure 17.2, bottom row):

- **Biconcave lens**: both surfaces concave (curved inwards); their radii may or may not be equal.
- **Plano-concave lens**: one surface flat, the other concave.
- **Convexo-concave lens**: one surface convex, the other concave, with the **concave surface curved more** (smaller radius) than the convex one, so the lens is thinner in the middle.

14. a. A converging lens is dipped in a liquid whose refractive index is greater than that of the lens. Does its power change? Explain.
    b. A convex lens is immersed in water. Will its focal length change?
    c. An object is placed 20 cm from a convex lens of focal length 10 cm. Where will the image be formed?

15. a. Can a convex and a concave lens be combined so that the combination behaves like a diverging lens? Justify your answer.
    b. If a glass lens is immersed in water, what happens to its power? Explain.
    c. A lens of focal length 12 cm produces a virtual image one-third the size of the object. What kind of lens is it? Find the positions of the object and the image.

16. a. If a converging lens and a diverging lens of the same focal length are in contact, how does the combination behave?
    b. At what distance from a convex lens should an object be placed on its axis to form a real image of the same size? Illustrate your answer with a ray diagram.
    c. What are the centers and radii of curvature of a lens?
    d. A concavo-convex lens has faces of radii of curvature 15 cm (the convex face) and 30 cm (the concave face). Find its focal length if the refractive index of its material is 1.6.

Numerical problems

17. A lens has a power of +5 D in air. What is its power when it is completely immersed in water? (Refractive index of glass 1.5, of water 1.33.)

18. A converging lens of focal length 5 cm is in contact with a diverging lens of focal length 10 cm. Find the focal length of the combination. Is the combination converging or diverging?

19. The radius of curvature of the convex face of a plano-convex lens is 12 cm, and the refractive index of its material is 1.5. Find its focal length.

20. A lens placed 20 cm from an object produces a virtual image two-thirds the size of the object. Find the position of the image, the kind of lens and its focal length.

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

Properties of spherical lenses

How a lens behaves depends on a few properties, which designers choose for each use (focus, clear images, brightness):

- **Focal length (f)**: the distance from the optical center to the principal focus. It sets how strongly the lens converges or diverges light: a lens with a **shorter focal length** converges (or diverges) **more strongly**; a longer focal length, more weakly.
- **Optical power (P)**: a measure of how strongly a lens bends (converges or diverges) light, the reciprocal of its focal length, $P = \frac{1}{f}$, with f in meters (17.6). A lens of higher power bends light more strongly.
- **Shape**: one or both faces of a spherical lens are parts of spheres, curved outwards (convex) or inwards (concave); the shape decides whether it converges or diverges and, with the glass, its focal length.
- **Aperture**: the effective diameter of the lens through which light passes. A wider aperture lets in more light and gives a brighter image (and, as Unit 18 shows, more blurring at the edges).

Terminology of a lens

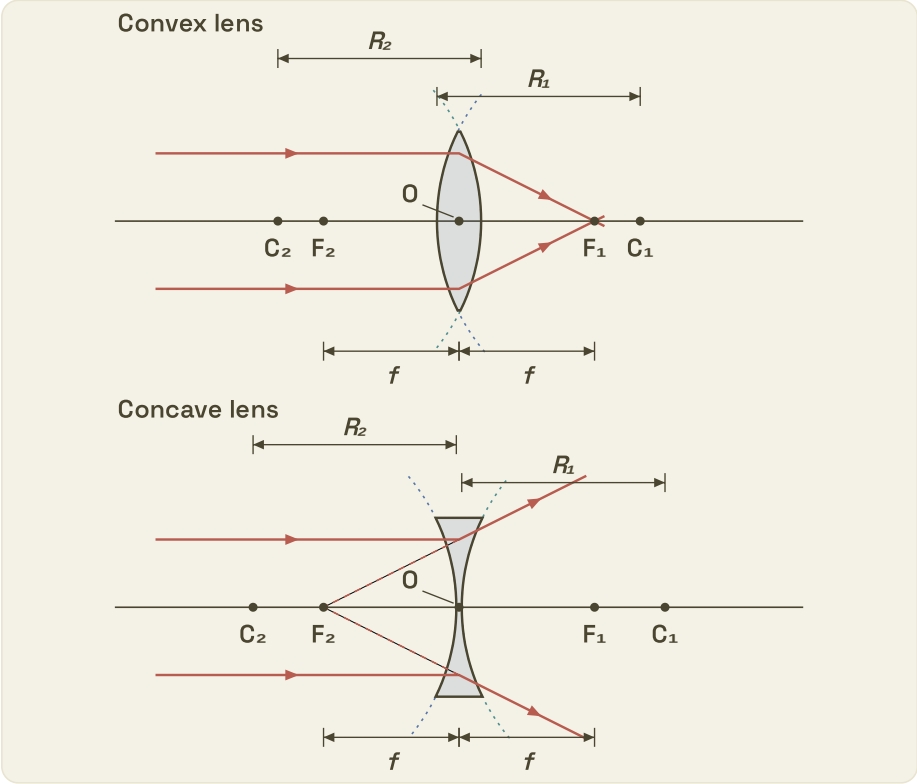


Figure 17.3 *A convex and a concave lens, cut through the middle. Each face is part of a sphere (dotted): C_1 and C_2 are the centers of those spheres, and R_1 and R_2 their radii. O is the optical center; F_1 and F_2 are the principal foci, a focal length f from O on either side. Drawn to scale for a dense glass of refractive index 1.75 (for ordinary glass, 1.5, an equiconvex lens's foci would be at its centers of curvature).*

- **Centers of curvature (C_1, C_2)**: a lens's two curved surfaces are parts of two spheres; the centers of those spheres are its **centers of curvature**. A lens has two.

- **Radii of curvature (R_1, R_2)**: the radii of those two spheres. The radii of a convex lens are taken as **positive**, those of a concave lens as **negative** (more exactly, below: a surface bulging outwards is positive, a surface curved inwards negative).
- **Principal axis**: the straight line through the two centers of curvature, C_1C_2. (For a lens with one flat face, the line through the center of curvature of the curved face, perpendicular to the flat face.)
- **Optical center (O)**: the point on the principal axis at the middle of a thin lens. A ray passing through it goes on **without being deviated** (in a thin lens it is barely shifted sideways, as by a thin slab of glass).
- **Principal focus (F)**: for a convex lens, the point on the principal axis where rays parallel to the axis **meet** after refraction; for a concave lens, the point from which they **appear to come** after refraction. A lens has two principal foci, F_1 and F_2, one on each side, at equal distances for a thin lens in air. The foci of a convex lens are **real**; those of a concave lens are **virtual**.
- **Focal length (f)**: the distance from the optical center to a principal focus, $OF_1 = OF_2 = f$. It is taken as **positive for a convex lens** and **negative for a concave lens**.
- **Focal plane**: the plane through a principal focus, perpendicular to the principal axis. Parallel rays that are not parallel to the axis meet at a point in it (in the camera of your phone, the sensor sits in the focal plane).

Check 1

A lens is thinner in the middle than at its edges, and one of its faces bulges outwards. It is a:

a concavo-convex lens, and it converges light
b convexo-concave lens, and it diverges light
c plano-concave lens, and it diverges light
d biconvex lens, and it converges light

17.2 Sign conventions of lenses

To use one formula for every lens and every image, distances get signs. A **sign convention** is a set of rules for these signs. Grade 11 uses the **real-is-positive** convention, the same one as for mirrors in Unit 14:

1. The aperture of the lens is small (and the lens is thin), so only rays close to the axis are considered.
2. All distances are measured from the **optical center** of the lens.
3. Distances of **real** objects and **real** images are **positive**; distances of **virtual** objects and **virtual** images are **negative**.
4. A convex lens can form both real and virtual images of a real object; a concave lens forms only virtual images of a real object.
5. The focal length and the radii of curvature of a **convex** lens are **positive**; those of a **concave** lens are **negative**.

So for a convex lens f is positive, and for a concave lens f is negative. A real image (on the far side of the lens, which can be caught on a screen) has a positive v; a virtual image (on the same side as the object, seen only by looking through the lens) has a negative v.

CAREFUL

Your other books may use another convention

Many other books use the **Cartesian** convention: distances measured in the direction the light travels are positive, against it negative. There a real object's distance is negative, and the lens formula becomes $\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$. Both give the same images; what matters is to use one convention all the way through a problem, with its own formula.

Check 9

5. A converging lens is placed inside a cell filled with a liquid of refractive index 1.60, the cell's two walls flat and thin. The focal length of the lens in air is 20 cm and its material has a refractive index of 1.50. What is the focal length of the system?

a −100 cm
b −90 cm
c −80 cm
d 24 cm

Check 10

6. The linear magnification of an image is m. What is the magnification of its area?

a m
b √m
c m^2
d m^4

Check 11

7. Two thin lenses of focal lengths f_1 and f_2 are in contact on the same axis. What is the power of the equivalent single lens?

a $f_1 + f_2$
b $\frac{f_1 f_2}{f_1 + f_2}$
c $\frac{f_1 + f_2}{2}$
d $\frac{f_1 + f_2}{f_1 f_2}$

Check 12

8. How does a convex lens behave when its second face is silvered?

a as a convex lens
b as a concave mirror
c as a convex mirror
d as a plate

Check 13

9. A convex lens is placed on a plane mirror. What happens to the power of the system if the space between the lens and the mirror is filled with water?

a it remains constant
b it increases
c it decreases
d it may increase or decrease depending on the focal length of the convex lens

Check 14

10. A convex lens is dipped in a liquid whose refractive index equals that of the lens. What happens to its focal length?

a it remains unchanged
b it becomes zero
c it becomes small but not zero
d it becomes infinite

Check 15

11. A convex lens of focal length f_1 is placed in contact with a concave lens of focal length f_2 (in size). In which condition does the combination act as a converging lens?

a $f_1 < f_2$
b $f_1 = f_2$
c $f_1 > f_2$
d $f_1 < 2f_2$

IN SHORT

Key ideas of Unit 17

- A **convex** lens (thicker in the middle) converges light; a **concave** lens (thinner in the middle) diverges it. Each has three shapes: bi-, plano- and the meniscus (concavo-convex, convexo-concave).
- **Real is positive**: distances of real objects and images positive, virtual negative; f and R positive for a convex lens, negative for a concave one.
- **Lens formula** $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$; **magnification** $m=\frac{I}{O}=\frac{v}{u}$.
- **Lens maker's formula** $\frac{1}{f}=(\mu-1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$, with μ relative to the medium around the lens: in water a glass lens weakens; in a denser liquid it diverges; an air bubble in water diverges.
- Lenses in contact: $\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$; **power** $P=\frac{1}{f}$ (diopters, f in m) adds: $P=P_1+P_2+\dots$
- The size of the retinal image is proportional to the **visual angle**; **angular magnification** $M=\beta/\alpha$.

Exercise

Choose the correct option

Check 5

1. Two lenses of power +12 D and −2 D are placed in contact. What is the focal length of the combination?

a 8.33 cm
b 10.00 cm
c 12.50 cm
d 16.60 cm

Check 6

2. If an object is placed at the focus of a convex lens, the refracted rays are:

a parallel
b converging
c diverging
d both converging and diverging

Check 7

3. What is the minimum distance between an object and its real image of the same size formed by a thin convex lens of focal length f?

a f/2
b f
c 2f
d 4f

Check 8

4. A convex lens of focal length 16 cm forms a virtual image twice the size of the object. What is the object distance?

a 8 cm
b 16 cm
c 24 cm
d 32 cm

Answer these questions

12.

a. When does a lens have the higher power: (i) in air or (ii) in water?

b. Why is the power of a lens measured as the reciprocal of its focal length?

c. What is the effective power, in diopters, of lenses of focal lengths +10 cm and −20 cm in contact?

17.3 Lens formula

The **lens formula** relates the object distance u, the image distance v and the focal length f of a lens, all measured from its optical center:

$$\boxed{\frac{1}{f}=\frac{1}{u}+\frac{1}{v}}$$

With the signs of 17.2, it holds for both convex and concave lenses and for real and virtual images: it gives where the image is, whether it is real or virtual, and, if u and v are measured, the focal length.

Deriving the lens formula

Take the case of a convex lens forming a real image (Figure 17.4).

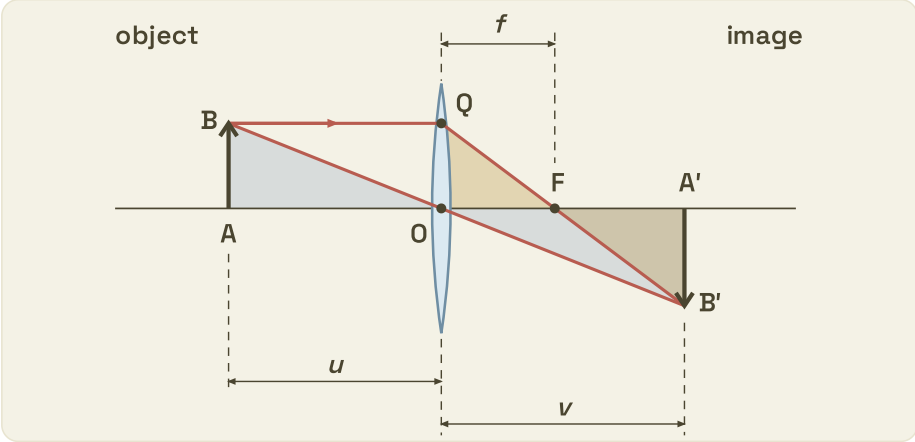


Figure 17.4 *An object AB on the axis of a convex lens and its real, inverted image A′B′. The ray from B parallel to the axis meets the lens at Q and passes through F; the ray through O goes straight on. Blue: the similar triangles ABO and A′B′O. Amber: the similar triangles OQF and A′B′F.*

Let AO = u (object distance), A′O = v (image distance) and OF = f (focal length). The ray from B parallel to the axis meets the lens at Q, so OQ = AB.

Similar triangles at the center. Triangles ABO and A′B′O are similar (right angles at A and A′, and the angles at O are vertically opposite):

$$\frac{A'B'}{AB}=\frac{A'O}{AO}\quad\dots(1)$$

Similar triangles at the focus. Triangles OQF and A′B′F are similar (right angles at O and A′, vertically opposite angles at F):

$$\frac{A'B'}{OQ}=\frac{A'F}{OF},\quad\text{and since } OQ=AB,\quad\frac{A'B'}{AB}=\frac{A'F}{OF}\quad\dots(2)$$

Put the two together. Equating (1) and (2), and writing A′F = A′O − OF:

$$\frac{A'O}{AO}=\frac{A'O-OF}{OF}\quad\Rightarrow\quad\frac{v}{u}=\frac{v-f}{f}$$

Cross-multiply. $vf=uv-uf$, so $uv=vf+uf$. … (3)

Divide by uvf**.** Dividing every term by it,

$$\frac{uv}{uvf}=\frac{vf}{uvf}+\frac{uf}{uvf}\quad\Rightarrow\quad\frac{1}{f}=\frac{1}{u}+\frac{1}{v}\quad\dots(4)$$

This is the lens formula. For a virtual image, or a concave lens, the same steps with that case's diagram give the same formula once v (and, for the concave lens, f) is given its negative sign.

The **magnification** is $m = \dfrac{v}{u}$ (17.8). With these signs, a positive m means a real, inverted image, and a negative m a virtual, upright one.

WORKED EXAMPLE

A lens of power 5 D

The real image formed by a lens of power 5 D is three times the length of the object. Find the object and image distances.

Answer

$f = \dfrac{1}{P} = \dfrac{1}{5}$ m = 0.2 m = 20 cm (positive: a convex lens).

The image is real and three times as long: $m = \dfrac{v}{u} = 3$, so $v = 3u$.

Lens formula: $\dfrac{1}{0.2} = \dfrac{1}{u} + \dfrac{1}{3u} = \dfrac{4}{3u}$, so $5 = \dfrac{4}{3u}$ and $u = \dfrac{4}{15}$ m = **26.7 cm**; $v = 3u$ = **80 cm**, on the other side of the lens.

17.4 Lens maker's formula

The **lens maker's formula** relates the focal length of a lens to the refractive index of its material and the radii of curvature of its two surfaces. Lens makers use it to grind lenses of a chosen power from a glass of a known refractive index, and opticians in Nepal grind spectacle lenses to it every day.

Deriving it: a lens as a thin prism

Take a thin convex lens of focal length f, made of a material of refractive index μ.

The turn of the ray. A ray MP, parallel to the principal axis at a small height h above it, strikes the lens at P. After refraction it passes through the focus F: it has been turned through an angle δ (Figure 17.6). In the right-angled triangle with sides h and f,

$$\tan\delta = \frac{h}{f}, \quad \text{and for a small angle } \tan\delta \approx \delta, \text{ so } \delta = \frac{h}{f} \quad \ldots(1)$$

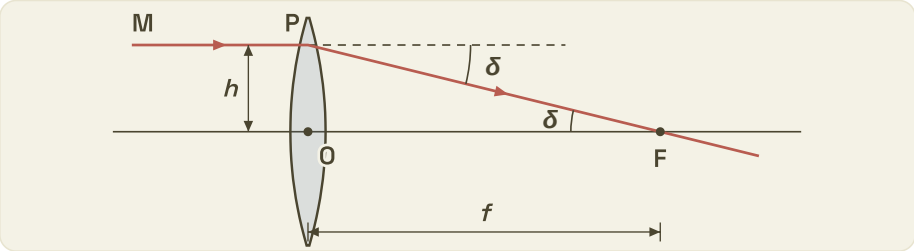


Figure 17.6 *A ray MP parallel to the axis, at a height h, is turned through δ by the lens and passes through the focus F: tan δ = h/f.*

The lens as a prism. Near the points P and Q where the ray enters and leaves, the two faces of the lens are like the two faces of a **thin prism** whose angle A is the angle between the tangents to the faces at P and Q (Figure 17.7). From Unit 16, a thin prism deviates a ray by

$$\delta = A(\mu - 1) \quad \ldots(2)$$

Equate the two. Equating (1) and (2):

larger (more than 1) or smaller (less than 1) the image is.

WORKED EXAMPLE

Finding a focal length by moving the lens

A luminous object and a screen are placed on an optical bench, and a converging lens between them forms a sharp image of the object on the screen, magnified 2.5 times. The lens is then moved 30 cm nearer the screen, and a sharp image is formed again. Find the focal length of the lens.

Answer

First position. $m = \dfrac{v}{u} = 2.5$, so $v = 2.5u$. … (1)

$\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{2.5u} = \dfrac{2.5+1}{2.5u} = \dfrac{1.4}{u}$ … (2)

Second position. The lens moves 30 cm towards the screen: the object distance becomes $u' = u + 30$ and the image distance $v' = v - 30 = 2.5u - 30$ (object and screen stay put).

$\dfrac{1}{f} = \dfrac{1}{u+30} + \dfrac{1}{2.5u - 30}$ … (3)

Solving. The quick way: with object and screen fixed, the lens law is the same if u and v swap, so the second sharp image comes when $u' = v$ and $v' = u$. Then $u + 30 = 2.5u$, so $1.5u = 30$ and $u = 20$ cm, $v = 50$ cm. (You can check that $u = 20$ satisfies (2) = (3): $\frac{1}{50} + \frac{1}{20} = \frac{7}{100}$.)

From (2): $\dfrac{1}{f} = \dfrac{1.4}{20}$, so $f = \dfrac{20}{1.4}$ = **14.3 cm**.

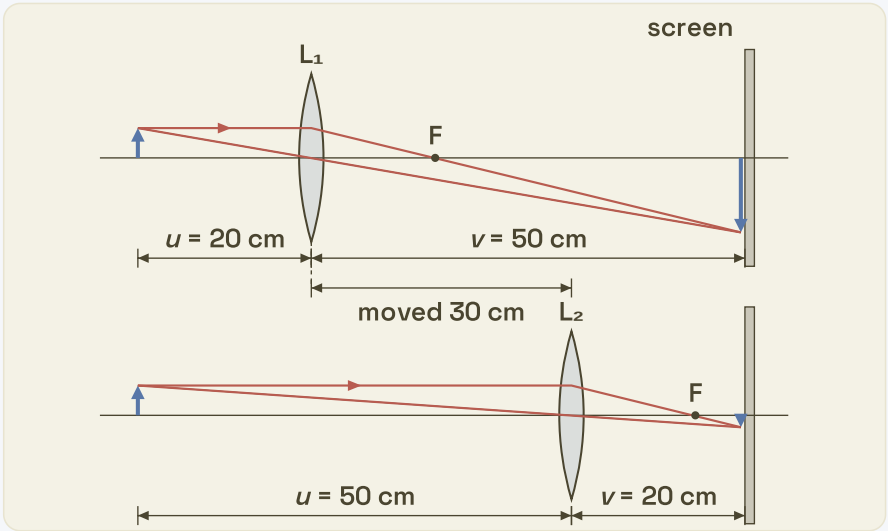


Figure 17.13 *To scale: object and screen 70 cm apart. With the lens 20 cm from the object, a sharp image 2.5 times as tall; moved 30 cm on, 50 cm from the object, a sharp image again, 0.4 times as tall. The two positions swap u and v.*

1. Angular magnification

The **angular magnification**, or **magnifying power**, of a lens is the ratio of the visual angle β subtended at the eye by the image to the visual angle α subtended at the eye by the object, when both are at the least distance of distinct vision D (about 25 cm for a normal eye, the nearest the eye can see clearly):

$$M = \frac{\beta}{\alpha}$$

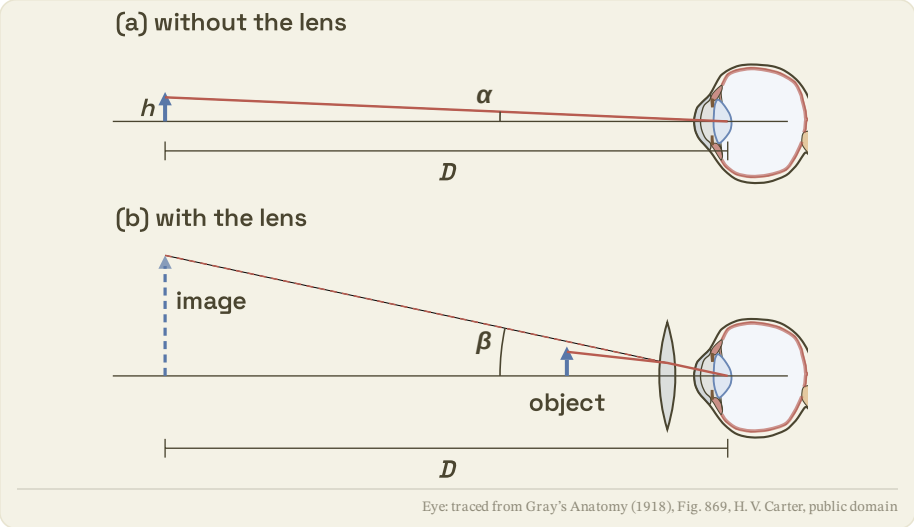


Figure 17.12 *A simple magnifier. (a) Without the lens, the object at the near point, a distance D from the eye, subtends α. (b) A convex lens just before the eye, with the object inside its focus, makes an upright, virtual image at D from the eye, which subtends the larger angle β. Here $f = D/4$ (the small gap between lens and eye neglected), so $M = 1 + D/f = 5$, and the image is five times as tall.*

The object can't be brought closer than D without blurring, so α is the largest angle the naked eye can give it. The lens lets you bring the object much closer (inside its focus) while the image stays at D, where the eye can see it clearly, filling a larger angle. (A simple magnifier that makes its image at D has $M = 1 + \dfrac{D}{f}$, with D = 25 cm: a lens of focal length 5 cm magnifies 6 times.)

2. Linear magnification

The **linear magnification** is the ratio of the size of the image to the size of the object:

$$m = \frac{\text{size of image (I)}}{\text{size of object (O)}} = \frac{I}{O} \quad \dots(1)$$

Consider an object AB and its image A′B′ formed by a convex lens with optical center C (Figure 17.4 shows the same picture, with O for C). The ray from B through the center C goes straight on to B′, so triangles ABC and A′B′C are similar:

$$\frac{A'B'}{AB} = \frac{A'C}{AC} = \frac{v}{u} \quad \dots(2)$$

From (1) and (2):

$$\boxed{m = \frac{I}{O} = \frac{A'B'}{AB} = \frac{v}{u}} \quad \dots(3)$$

So the linear magnification is also the ratio of the image distance to the object distance. With the signs of 17.2, m is positive for a real (inverted) image and negative for a virtual (upright) one; its size says how many times

$$\frac{h}{f} = A(\mu - 1) \quad \Rightarrow \quad \frac{1}{f} = \frac{A}{h}(\mu - 1) \quad \dots(3)$$

The prism's angle from the curves. Now find A from the faces' curves. Let C_1 and C_2 be the centers of curvature of the two faces, with radii R_1 and R_2 (for a thin lens, $C_1O = R_1$ and $C_2O = R_2$). The radius C_1P is perpendicular to the face at P, and C_2Q to the face at Q. Let K be the point of the lens at height h above O (KO = h), and let $\angle KC_2O = \theta$ and $\angle KC_1O = \varphi$. Since a tangent is perpendicular to its radius, the angle A between the two tangents equals the angle between the two radii, which is $\theta + \varphi$:

$$A = \theta + \varphi \quad \dots(4)$$

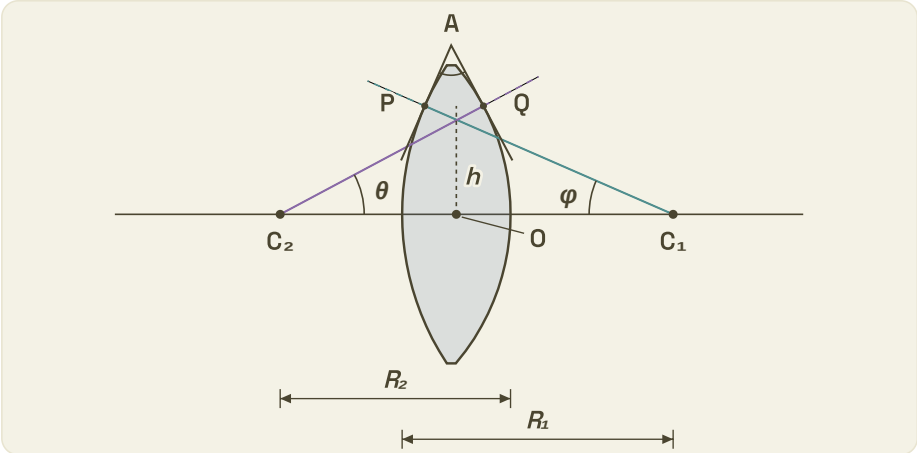


Figure 17.7 *Near P and Q the lens is a thin prism: the tangents to its faces meet at A. Each radius is perpendicular to its face, so the prism's angle A equals $\theta + \varphi$, the angles the radii C_2Q and C_1P make with the axis. Drawn exactly for faces of different radii.*

Small angles. For small angles, $\theta \approx \tan\theta = \dfrac{h}{R_2}$ and $\varphi \approx \tan\varphi = \dfrac{h}{R_1}$. So

$$A = \frac{h}{R_1} + \frac{h}{R_2} = h\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad \text{or} \quad \frac{1}{R_1} + \frac{1}{R_2} = \frac{A}{h} \quad \dots(5)$$

The result. Putting (5) into (3):

$$\boxed{\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)} \quad \dots(6)$$

This is the **lens maker's formula**, derived for a convex lens. The height h has canceled: every ray parallel to the axis, at any (small) height, is sent through the same point F, which is why a lens has a focus at all.

For a concave lens, both surfaces curve inwards: with the signs of 17.2, R_1 and R_2 are both negative, $-R_1$ and $-R_2$ written with their sizes, so

$$\frac{1}{f} = (\mu - 1)\left(-\frac{1}{R_1} - \frac{1}{R_2}\right) = -(\mu - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

and f comes out negative, as it must for a diverging lens.

For a lens of any shape, give each face's radius its own sign: **positive if the face bulges outwards** (convex), **negative if it curves inwards** (concave), and **infinite** (so $1/R = 0$) if it is flat. Here μ is the refractive index of the lens **relative to the medium around it**; in air, simply its refractive index.

CAREFUL

What the formula assumes

The lens is thin (its thickness is small compared with f and the radii), the rays are close to the axis (all the angles are small), and μ is for one color of light. Thick lenses, wide rays and white light all depart from it a little: those departures are the aberrations of Unit 18.

WORKED EXAMPLE

Sunglasses with curved faces but zero power

The surfaces of sunglasses are curved, yet their power is zero. How?

Answer

Both faces of the glass curve the same way, with their centers of curvature on the same side and (for zero power) equal radii, like a piece cut from a thin shell. One face bulges outwards and the other curves inwards, so the radii are equal in size and opposite in sign: $R_1 = R$, $R_2 = -R$.

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R} + \frac{1}{-R}\right) = (\mu - 1)\left(\frac{1}{R} - \frac{1}{R}\right) = 0$$

So f is infinite and the power $P = 1/f = 0$: the glasses neither converge nor diverge light. (Strictly, a real glass of some thickness has a tiny power; it is made negligible by choosing the radii a little differently.) Their curve only wraps them round the face.

WORKED EXAMPLE

A lens in water: more or less power?

A glass lens is immersed in water. Does its power increase or decrease?

Answer

Let the refractive index of the glass be ${}_a\mu_g$ relative to air and ${}_w\mu_g$ relative to water, and the radii R_1 and R_2.

In air: $P_1 = ({}_a\mu_g - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ … (1)

In water: $P_2 = ({}_w\mu_g - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ … (2)

Dividing (2) by (1):

$$\frac{P_2}{P_1} = \frac{{}_w\mu_g - 1}{{}_a\mu_g - 1}$$

Since ${}_w\mu_g = \frac{{}_a\mu_g}{{}_a\mu_w} = \frac{1.5}{1.33} = 1.13$ is smaller than ${}_a\mu_g = 1.5$, P_2 is smaller than P_1: the **power decreases** (to about a quarter for ordinary glass). Light is bent less at a glass–water surface than at a glass–air surface, because the two media are more alike. This is why you can't see clearly under water: your cornea, whose job is to bend light, loses nearly all its power there; goggles put the air back in front of it.

Check 4

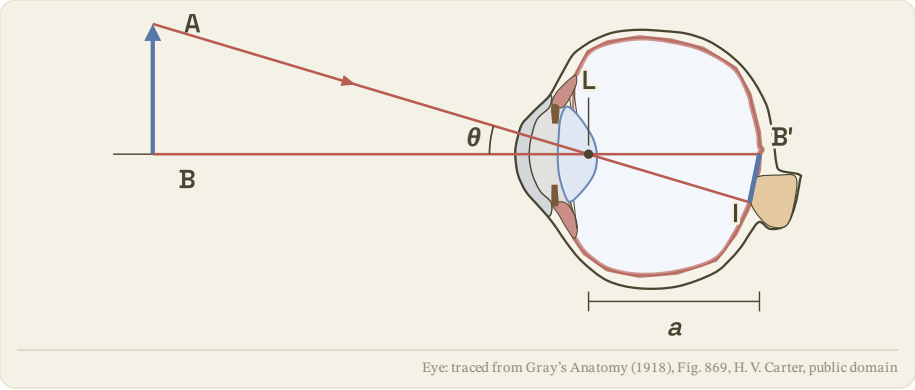
A lens of power −4 D is placed in contact with a lens of power +2.5 D. The pair is:

a converging, focal length 66.7 cm
b diverging, focal length −66.7 cm
c diverging, focal length −15 cm
d converging, focal length 15 cm

17.7 Visual angle

Why does a coin held close to your eye hide a whole mountain? Because what decides how big a thing **looks** is not its real size but the angle it fills at your eye.

The **visual angle** is the angle that an object (or an image) subtends at the eye: the angle between the two lines from the ends of the object to the **nodal point** of the eye (Figure 17.11). (The nodal point is the point inside the eye through which rays pass on without being bent; in a simplified eye it is about 7 mm behind the front of the cornea.) It depends on two things: the **size** of the object and its **distance** from the eye.



Eye: traced from Gray's Anatomy (1918), Fig. 869, H. V. Carter, public domain

Figure 17.11 *The visual angle: lines from the ends A and B of an object through the eye's nodal point L reach the retina at I and B′. The object subtends the angle θ at L, and its image on the retina is IB′ = θ × a long, where a is the distance from L to the retina. The eye is drawn to scale (Gray's Anatomy).*

In Figure 17.11, the object AB has height h and subtends the visual angle θ at the nodal point L. Its image IB′ on the retina has height h'. Because θ is small, the image is very nearly an arc of a circle of radius LB′ = a, the distance from the nodal point to the retina, so

$$h' = IB' = \theta \times a \quad (\theta \text{ in radians})$$

The distance a is fixed for a given eye (about 17 mm), so

$$h' \propto \theta$$

The **size of the image on the retina is proportional to the visual angle**: the larger the visual angle, the larger the image and the larger the object looks. A 1.6 m friend 160 m away and a 1 cm coin 1 m away subtend the same angle (0.01 rad), and look the same size. That is what every magnifying instrument does: it increases the visual angle.

17.8 Magnification produced by a lens

Magnification is the ratio of the size of an image to the size of the object. Lenses, microscopes and telescopes all magnify, and how much a lens magnifies depends on where the object is. There are two kinds of magnification.

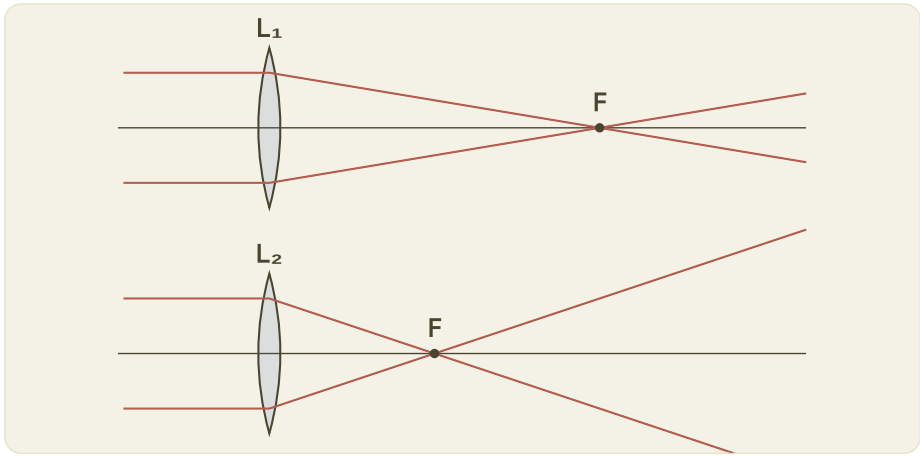


Figure 17.10 *The power of a lens. The same parallel rays fall on two convex lenses. L_2 has half the focal length of L_1, so it turns each ray through about twice the angle and brings the rays to a focus half as far away: it has twice the power.*

That is why opticians use powers rather than focal lengths: to correct an eye, they add the lens's power to the eye's own. The equivalent power of lenses in contact is the **algebraic** sum of their powers, each with its sign.

If a convex lens and a concave lens of **equal power** are combined, $P = P_1 + (-P_1) = 0$: the pair has zero power and behaves like a flat **glass plate**, neither converging nor diverging light.

WORKED EXAMPLE

A converging and a diverging lens of the same focal length

How does the combination behave if a converging lens and a diverging lens of the same focal length are placed in contact?

Answer

The diverging lens has the negative of the converging lens's focal length: $f_1 = -f_2$. Then

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} = -\frac{1}{f_2} + \frac{1}{f_2} = 0$$

So $\frac{1}{F} = 0$: the focal length is **infinite** and the power $P = \frac{1}{F} = 0$. The pair acts as a plain glass plate (or a watch glass): parallel light comes out parallel.

DID YOU KNOW?

Nepal's lens factory

The intraocular lenses that replace a clouded eye lens (a cataract) once cost hundreds of dollars each. In the 1990s the Fred Hollows Intraocular Lens Laboratory at the Tilganga Institute of Ophthalmology in Kathmandu, set up with the eye surgeon **Sanduk Ruit**, began making them for a few dollars. The lab has since made millions, used in Nepal and sent to dozens of countries, and Dr. Ruit's quick, stitchless cataract operation has restored the sight of well over a hundred thousand people. Each of those tiny lenses is ground to a power worked out for one patient's eye, typically about +20 D.

WORKED EXAMPLE

An air bubble in water

An air bubble is formed in water. Does it act as a converging lens or a diverging lens?

Answer

A bubble is a biconvex "lens" of air, and the medium around it is water. In the lens maker's formula, μ is the refractive index of air relative to water:

$$\mu = {}_w\mu_a = \frac{1}{1.33} = 0.75 < 1, \quad \text{so} \quad \frac{1}{f} = (0.75 - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) < 0.$$

The focal length is negative: the bubble acts as a **diverging lens**, even though it is convex in shape (Figure 17.8). Light going from water into air bends away from the normal, the opposite of light entering glass from air.

Bubbles in water often shine like silver, though. That is not refraction: light striking a bubble's surface from the water at a large angle of incidence (more than the critical angle, 48.8°) is **totally internally reflected** (Unit 15).

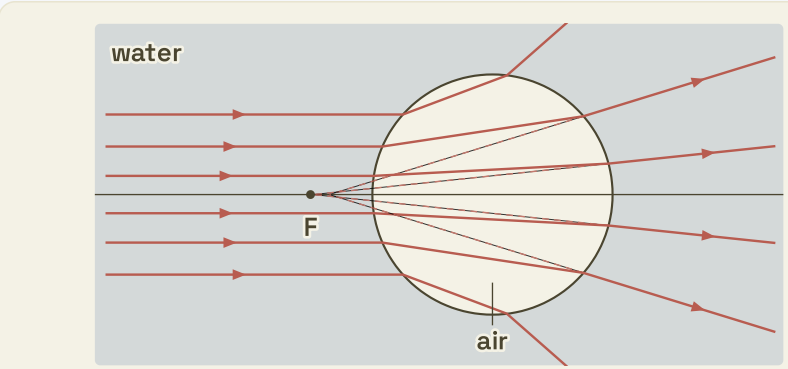


Figure 17.8 *Rays parallel to the axis through an air bubble in water, traced exactly by Snell's law: they leave spreading out, as if from a point F before the bubble. A convex bubble of air is a diverging lens.*

WORKED EXAMPLE

A convex lens in water

A convex lens of focal length 24 cm in air, made of glass of refractive index 1.5, is totally immersed in water (refractive index 1.33). Find its focal length in water.

Answer

In air: $\frac{1}{24} = (1.5 - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$, so $\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{0.5 \times 24} = \frac{1}{12}$ cm^{-1}. … (1)

In water, the glass's refractive index relative to water is ${}_w\mu_g = \frac{{}_a\mu_g}{{}_a\mu_w} = \frac{1.5}{1.33} = 1.128$. So

$$\frac{1}{f_1} = ({}_w\mu_g - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = (1.128 - 1) \times \frac{1}{12} = \frac{0.128}{12} = 0.01065 \text{ cm}^{-1}$$

$f_1 =$ **93.9 cm**, nearly four times its focal length in air.

WORKED EXAMPLE

The radii of an equiconvex lens

An equiconvex lens has a power of 4 D. Find the radius of curvature of each face if its glass has a refractive index of 1.5.

Answer

Equiconvex: $R_1 = R_2 = R$. So $P = \dfrac{1}{f} = (\mu - 1)\left(\dfrac{1}{R} + \dfrac{1}{R}\right) = \dfrac{2(\mu - 1)}{R}$.

$R = \dfrac{2(\mu - 1)}{P} = \dfrac{2 \times 0.5}{4} = 0.25 \text{ m} = \mathbf{25\ cm}$ each.

(For glass of 1.5, an equiconvex lens's radius equals its focal length, $f = R$.)

DID YOU KNOW?

A glass rod that disappears

Put a glass rod or a Pyrex test tube into a beaker of vegetable oil, or of glycerine: much of it seems to vanish. When the liquid's refractive index equals the glass's, light crosses the boundary without bending or reflecting, so there is nothing to see. In the lens maker's formula, $\mu - 1 = 0$: no power at all. Liquids matched to a glass this way are used to find its refractive index.

Check 2

A plano-convex lens is made of glass of refractive index 1.5, and its curved face has a radius of curvature of 10 cm. Its focal length in air is:

a 5 cm b 10 cm
c 20 cm d 40 cm

17.5 Combination of two lenses in contact

Lenses are often used in groups: in a camera, a microscope or a pair of binoculars, several lenses act together as one. Take two thin lenses L_1 and L_2, of focal lengths f_1 and f_2, placed in contact on a common axis (Figure 17.9). Here both are convex; the result holds for any pair (both convex, both concave, or one of each), with each focal length given its sign.

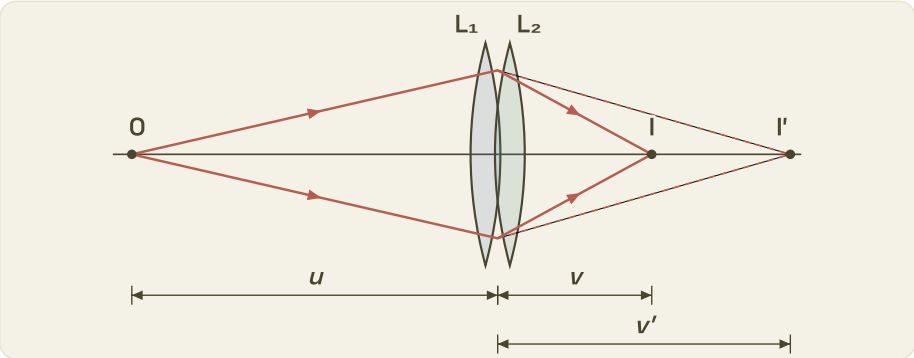


Figure 17.9 *Two thin convex lenses L_1 and L_2 in contact. On its own, L_1 would form the image of O at I′ (dashed); L_2 bends the converging rays further, to I. Drawn to scale for $f_1 = 120$, $f_2 = 240$ and $u = 270$ drawing units.*

The first lens. A point object O at a distance u from the lenses. Lens L_1 alone would form its image I′ at a distance v' on the other side:

$$\frac{1}{f_1} = \frac{1}{u} + \frac{1}{v'} \quad \dots(1)$$

The second lens. But L_2 is in the way. The rays reaching L_2 are converging towards I′, so for L_2, I′ is a **virtual object**, at a distance $-v'$ (negative, by the sign convention). L_2 forms the final image I at a distance v:

$$\frac{1}{f_2} = \frac{1}{-v'} + \frac{1}{v} \quad \dots(2)$$

Add them. Add (1) and (2); the v' terms cancel:

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{u} + \frac{1}{v} \quad \dots(3)$$

One lens in their place. If the pair is replaced by a single lens of focal length F that forms the same image I of O, then

$$\frac{1}{F} = \frac{1}{u} + \frac{1}{v} \quad \dots(4)$$

Compare. Comparing (3) and (4):

$$\boxed{\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}} \quad \dots(5)$$

This is the focal length of two thin lenses in contact. For more lenses in contact, $\dfrac{1}{F} = \dfrac{1}{f_1} + \dfrac{1}{f_2} + \dfrac{1}{f_3} + \dots$

Check 3

Two thin convex lenses of focal lengths 20 cm and 30 cm are placed in contact. The focal length of the pair is:

a 50 cm b 25 cm
c 12 cm d 10 cm

17.6 Power of a lens

Do you know why some people in your neighborhood wear spectacles, and what power their lenses have? An optician's prescription says it in **diopters**.

Every ray falling on a convex lens is bent towards the principal axis and tends to converge; every ray falling on a concave lens is bent away from the axis and tends to diverge. A lens's job is to bend light, so its **power** is measured by how strongly it does so.

The **power** of a lens is its ability to converge or diverge the light passing through it. It is defined as the **reciprocal of its focal length**:

$$P = \frac{1}{f} \quad (f \text{ in meters})$$

A lens of shorter focal length bends the rays more: it brings parallel rays to a nearer focus. (Raise the liquid's index in the widget of 17.4 and watch the power fall as the focus moves away.)

The unit of power is the **diopter** (D), with the focal length in meters: a lens of focal length 1 m has a power of 1 D, and one of 25 cm, $\frac{1}{0.25} = 4$ D. A **convex** lens bends rays towards the axis, and its power is **positive**; a **concave** lens bends them away, and its power is **negative**. A prescription of -2.5 D is a concave lens of focal length -40 cm, for short sight.

For lenses in contact, from 17.5, $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \dots$, so the **powers add**:

$$\boxed{P = P_1 + P_2 + P_3 + \dots}$$