Unit 16

Refraction through Prisms

This unit online, with its interactives:
science-playground.com/s/11f16

Hold a glass prism to a window and the room beyond it jumps sideways, and its edges grow fringes of color. A prism is only a wedge of glass, yet with it Newton split sunlight into a rainbow, and with it laboratories today measure refractive indices to six decimal places.

This unit follows one ray through a prism, face by face, using nothing but Snell's law from Unit 15. You will find how much the prism turns the ray, why there is one angle at which it turns the ray least, and how that least turning gives the refractive index of the glass. Then you will see why a thin prism turns every ray by the same amount, the idea behind the prism glasses an eye doctor prescribes.

A STORY FROM SCIENCE

The mathematician who wrote the colors into an equation

Augustin-Louis Cauchy (1789–1857) was born in Paris a month after the storming of the Bastille, and became one of the greatest mathematicians of his century. He made calculus rigorous: the definitions of limits and continuity still taught today are essentially his. In 1816 he won the grand prize of the French Academy of Sciences for a paper on waves on the surface of water, and in 1822 he laid the foundations of the mathematical theory of elasticity, how solids stretch and bend (Unit 8).

In optics, in 1836–37 he gave a simple relation between the refractive index of a transparent material and the wavelength of light, now called **Cauchy's equation**: $\mu = A + B/\lambda^2$, with constants A and B for each material. It says that shorter wavelengths (violet) are bent more than longer ones (red), which is why a prism spreads white light into colors (Unit 18). Cauchy wrote so much that the Academy limited the length of papers in its journal, partly because of him. His collected works fill 27 volumes, published from 1882 to 1974.



Figure 16.1 *Augustin-Louis Cauchy (1789–1857).*
Photo: Public domain, Public domain

16.1 Introduction

A **prism** is a block of transparent material, usually glass, with two plane faces inclined to each other. A prism looks simple, but it opened the way to many discoveries about light: Newton's splitting of white light

into colors, the spectra that tell us what stars are made of, and many of the instruments of optics, from binoculars to spectrometers. In this unit we study, in detail, how light is refracted through a prism.

A prism is a homogeneous, transparent medium enclosed by two plane surfaces that meet along a straight edge. The two surfaces through which light passes are the **refracting faces**; the line where they meet is the **refracting edge**; and the angle between them is the **angle of the prism** or **refracting angle**, A. The face opposite the edge is the **base**. A plane cut through the prism at right angles to its refracting edge gives a triangle, the **principal section** of the prism; in this unit all rays travel in such a plane.

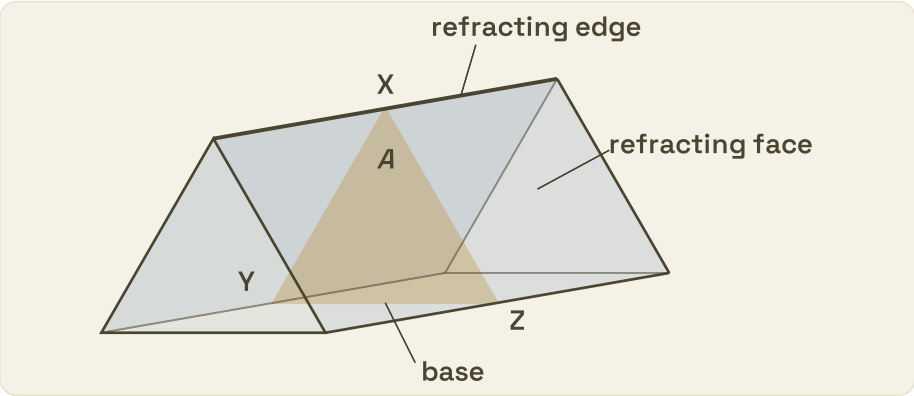


Figure 16.2 *A glass prism. The two sloping faces are its refracting faces, which meet along the refracting edge; the angle between them, A, is the angle of the prism. The shaded triangle XYZ, cut at right angles to the refracting edge, is the principal section.*

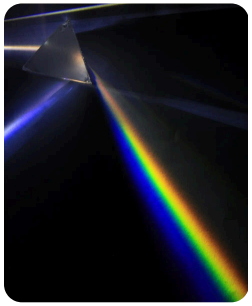


Figure 16.3 *Light through a prism of flint glass, seen end-on, so that its principal section shows as a triangle. Each color is turned towards the base by a slightly different amount, so the light comes out spread into separate colors (Unit 18).*
Photo: D-Kuru, CC BY-SA 3.0 at

16.2 Refraction through a prism

Take a prism XYZ with refracting angle A at X, and a ray of light of one color, PQ, falling on face XY at Q (Figure 16.4). It is refracted into the glass along QR, towards the normal, meets the second face XZ at R, and is refracted again, away from the normal, leaving along RS. MN and ON are the normals at Q and R; they meet at N.

14. a. Does the angle of deviation depend on the angle of the prism?
    b. List the conditions for minimum deviation through a prism.
    c. Light falls on one face of a 60° prism of refractive index 1.5 at an angle of incidence of 90° (Figure 16.5). Find the angle of emergence and the angle of deviation.

Numerical problems

15. Calculate the angle of minimum deviation for light passing through a prism of angle 60° and refractive index 1.5. What is the angle of incidence for it?

16. A ray of light is refracted through a prism at minimum deviation. The angle of the prism is 60° and the deviation is 34°. Find the angle of incidence and the refractive index.

17. A ray of light falls normally on one face of a glass prism of refractive index 1.5. Find the angle of the prism if the ray just fails to come out of the second face.

18. What should the refracting angle of a prism of refractive index 1.5 be, so that a ray falling on a face at 50° passes at minimum deviation?

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

- i is the **angle of incidence** at the first face, between PQ and the normal.
- r_1 is the **angle of refraction** at the first face.
- r_2 is the angle of incidence inside the glass at the second face.
- e is the **angle of emergence**, between the emergent ray RS and the normal at R.
- δ is the **angle of deviation**, the angle between the direction of the incident ray and that of the emergent ray. Produce PQ forwards and SR backwards; they meet at T, and δ is the angle between them there.

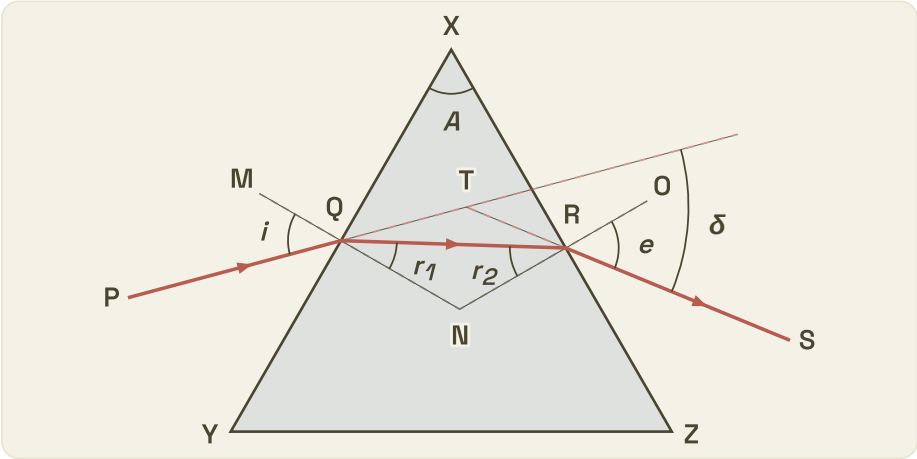


Figure 16.4 *A ray through a prism of glass ($\mu = 1.5$), traced by Snell's law: $i = 45^\circ$, $r1 = 28.1^\circ$, $r2 = 31.9^\circ$, $e = 52.4^\circ$. The ray is turned towards the base twice, at Q and at R; produced, the incident and emergent rays meet at T, where the deviation, $\delta = 37.4^\circ$, is marked.*

The angle of the prism. In the four-sided figure XQNR, the angles at Q and R are right angles (the normals are perpendicular to the faces), and the four angles add up to 360°. So

$$A + \angle QNR = 180^\circ.$$

In the triangle QNR, the angles at Q and R are r_1 and r_2, so

$$r_1 + r_2 + \angle QNR = 180^\circ.$$

Comparing the two:

$$r_1 + r_2 = A \tag{16.1}$$

The deviation. At Q the ray is turned through $i - r_1$; at R it is turned again, the same way, through $e - r_2$. The total turning is

$$\delta = (i - r_1) + (e - r_2) = (i + e) - (r_1 + r_2)$$

$$\delta = i + e - A \tag{16.2}$$

(In the triangle QTR the exterior angle at T is the sum of the interior angles at Q and R, which are $i - r_1$ and $e - r_2$: the same result.) Together with Snell's law at each face, $\sin i = \mu \sin r_1$ and $\sin e = \mu \sin r_2$, these two equations let us follow any ray through any prism.

A prism of glass in air always turns light, overall, **towards its base**; for most rays, as in Figure 16.4, both faces turn it that way. The faces of a glass slab are parallel, so the turning at the second face undoes the first, and the light only shifts sideways (Unit 15); the faces of a prism lean towards each other, so the second turning never undoes the first.

CAREFUL

Ray in, ray out, for one color

These equations are for light of one color, which has one refractive index. White light is a mixture of colors, each with a slightly different μ, so each color leaves at a slightly different angle: that is dispersion, the subject of Unit 18. Here we follow one color at a time.

WORKED EXAMPLE

Example 1: Light grazing the first face

Light falls on one face of a 60° prism of refractive index 1.5 at an angle of incidence of 90°, grazing the face. Find the angle of emergence and the angle of deviation.

Answer

At grazing incidence the ray inside the glass is refracted at the critical angle:

$$\sin r_1 = \frac{\sin 90^\circ}{1.5} = 0.6667 \;\Rightarrow\; r_1 = 41.8^\circ$$

At the second face, from $r_1 + r_2 = A$:

$$r_2 = 60^\circ - 41.8^\circ = 18.2^\circ$$

$$\sin e = 1.5 \sin 18.2^\circ = 1.5 \times 0.312 = 0.469 \;\Rightarrow\; e = 27.9^\circ$$

The deviation:

$$\delta = i + e - A = 90^\circ + 27.9^\circ - 60^\circ = 57.9^\circ$$

The angle of emergence is **27.9°** and the deviation **57.9°**.

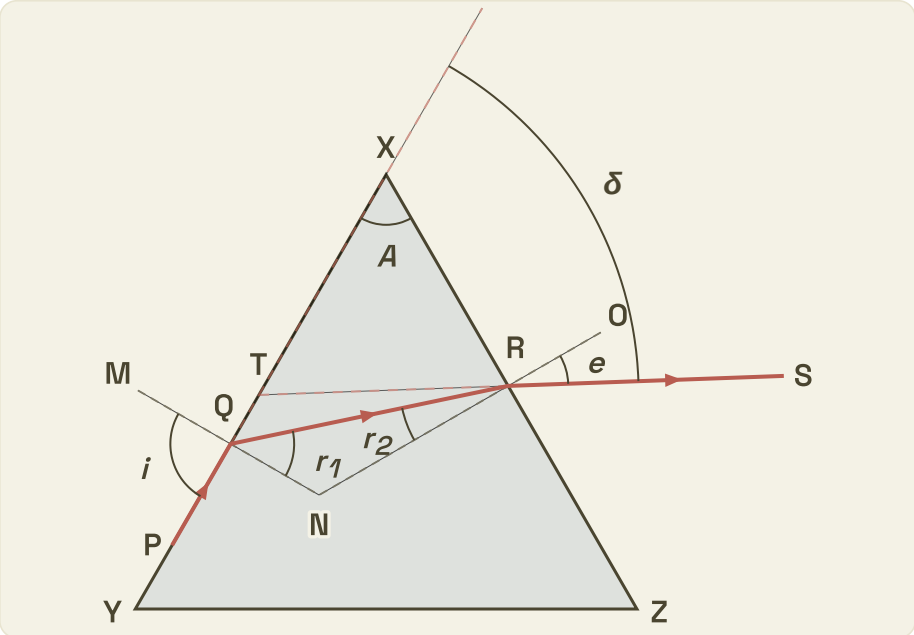


Figure 16.5 *Example 1: light grazing the first face of a 60° glass prism enters at the critical angle, 41.8°, and leaves at 27.9°.*

Check 11

7. The refracting angle of a prism is A and its refractive index is $\cot(A/2)$. What is its angle of minimum deviation?

a $180^\circ - 3A$ b $180^\circ - 2A$
c $90^\circ - A$ d $180^\circ + 2A$

10.
a. Why don't we speak of a lateral shift for a prism?
b. Light falls normally on one face of a prism and leaves the second face grazing it (at 90° to the normal). What is the relation between the angle of the prism and the critical angle?
c. A ray of light makes an angle of 50° with one face of a prism. The angle of the prism is 60° and the ray is deviated through 42°. Find the angle which the emergent ray makes with the second face.

11.
a. State the factors on which the deviation produced by a prism depends.
b. A ray of light falls at an angle of incidence of 60° on one face of a prism whose refracting angle is 30°. The angle between the emergent ray and the incident ray is 30°. What is the angle between the emergent ray and the face from which it emerges?

12.
a. A glass prism is put in water. What happens to its angle of minimum deviation?
b. Light falls normally on one of the shorter faces of a right-angled isosceles prism and is totally reflected at the long face. What is the least refractive index the prism can have?

13.
a. Does the angle of deviation depend on the angle of the prism? Explain.
b. In an equilateral prism the angle of incidence equals the angle of emergence, and each is three-quarters of the angle of the prism. Find the angle of deviation.

Exercise

Choose the correct option

Check 5

1. Which of these must equal the angle of emergence when light passes through a prism at minimum deviation?

a the angle of incidence
b the angle of the prism
c the angle of refraction
d the angle of deviation

Check 6

2. Which of these is false for a prism placed at minimum deviation?

a $i = e$
b $r_1 = r_2$
c $i = r_1$
d Light passes symmetrically through the prism.

Check 7

3. The critical angle for the material of a prism is C, and the angle of the prism is A. In which case is there no emergent ray, whatever the angle of incidence?

a $A < 2C$
b $A = 2C$
c $A > 2C$
d $A < C$

Check 8

4. What should the refracting angle of a prism of refractive index 1.5 be, for a ray falling on a face at 50° to pass at minimum deviation?

a 59.42°
b 60.42°
c 61.42°
d 70.42°

Check 9

5. An equilateral prism has a refractive index of $\sqrt{2}$. At what angle of incidence does minimum deviation happen?

a 60°
b 45°
c 30°
d 20°

Check 10

6. The angle of a prism is A and its angle of minimum deviation is $180^\circ - 2A$. What is the refractive index of the prism?

a $\sin\frac{A}{2}$
b $\cos\frac{A}{2}$
c $\tan\frac{A}{2}$
d $\cot\frac{A}{2}$

Answer these questions

8. a. Can a ray of light ever pass through a prism without being deviated? Explain.

b. One face of a prism of refractive index $\sqrt{2}$ and refracting angle 30° is silvered. At what angle must a ray fall on the other face so that, after refraction there and reflection at the silvered face,

YOUR TURN

Try it

1. Why don't we speak of a lateral shift for a prism, as we do for a glass slab?
2. A ray of light falls on a glass prism at an angle of incidence of 40°. The angle of the prism is 60° and its refractive index 1.5. Find the angle of emergence.
3. An equilateral prism of refractive index 1.45 is under water (refractive index 1.33). A narrow beam of light falls normally on one face. Find the direction of the emergent ray.

Answer

1. A lateral shift is the distance between two parallel rays. Light leaving a slab is parallel to the light that went in, so it has only moved sideways, and the shift measures that. Light leaving a prism is not parallel to the light that went in: it has been turned, and the angle it was turned through, the deviation, is what describes it.
2. $\sin r_1 = \sin 40^\circ / 1.5 = 0.4285$, so $r_1 = 25.4^\circ$. Then $r_2 = 60^\circ - 25.4^\circ = 34.6^\circ$, and $\sin e = 1.5 \sin 34.6^\circ = 0.852$, so $e =$ **58.5°** (and $\delta = 40^\circ + 58.5^\circ - 60^\circ = 38.5^\circ$).
3. The beam enters along the normal, so $i = 0$, $r_1 = 0$ and $r_2 = A = 60^\circ$. At the second face it goes from the prism into water, so the relative index $1.45/1.33 = 1.090$ counts: $1.45 \sin 60^\circ = 1.33 \sin e$, so $\sin e = \dfrac{1.45 \times 0.866}{1.33} = 0.944$ and $e = 70.8^\circ$. The beam leaves at **70.8°** to the normal of the second face, turned through $\delta = 0 + 70.8^\circ - 60^\circ =$ **10.8°** towards the base (in air it would not get out at all: $1.45 \sin 60^\circ = 1.26 > 1$). It leaves at $90^\circ - 70.8^\circ = 19.2^\circ$ to the face itself.

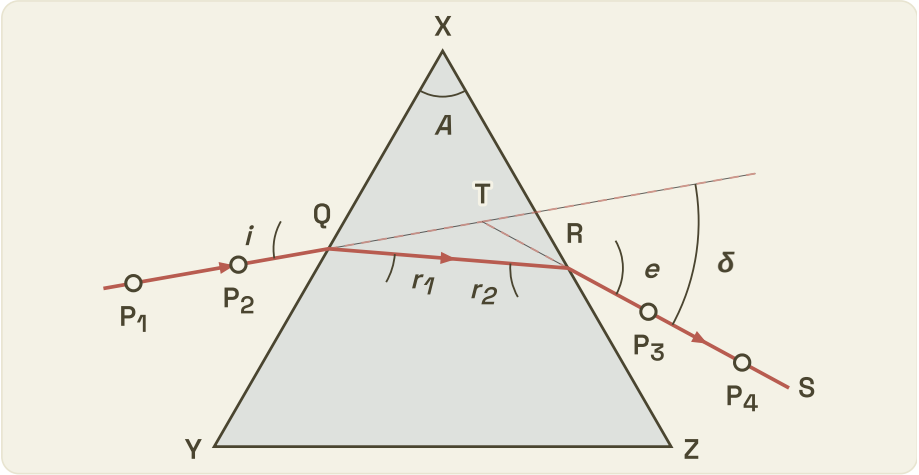


Figure 16.6 Activity 16.1: the pins P_1 and P_2 on the incident ray, and P_3 and P_4 placed so that they line up with the images of P_1 and P_2 seen through the prism. The line through P_3 and P_4 is the emergent ray.

Check 1

A ray passes through a prism of angle 60°. Its angle of refraction at the first face is 35°. What is its angle of incidence at the second face?

a 35°
b 25°
c 60°
d 95°

Check 2

A ray falls on a 60° prism at 50° and leaves at 40°. What is its deviation?

a 10° b 30°

c 90° d 150°

16.3 Angle of minimum deviation and refractive index

Turn the prism slowly while you watch a ray through it, and something surprising happens. As the angle of incidence grows, the deviation **falls**, reaches a smallest value, and then **rises** again. The graph of δ against i has a single lowest point; that least deviation is the **angle of minimum deviation**, δ_m.

At minimum deviation the ray passes through the prism **symmetrically**:

- the angle of incidence equals the angle of emergence, $i = e$;
- the angles inside the glass are equal, $r_1 = r_2 = r$;
- in a prism with equal sides about the edge, the ray inside runs **parallel to the base**.

Why it must be symmetrical. Light can retrace its path (Unit 15). Suppose the least deviation happened for a ray that went in at i and came out at e, with $i \neq e$. Send light backwards along that path: it goes in at e, comes out at i, and is turned through the same angle. So the deviation would be least at two different angles of incidence, i and e. But the graph has only one lowest point. The two must be the same: $i = e$, and then by Snell's law $r_1 = r_2$.

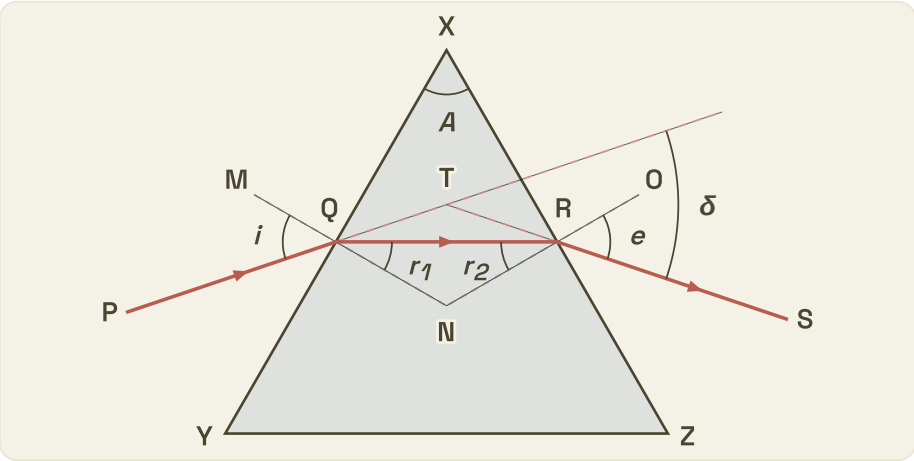


Figure 16.7 *Minimum deviation in a 60° glass prism ($\mu = 1.5$): $i = e = 48.6^\circ$, $r_1 = r_2 = 30^\circ$, and the ray inside runs parallel to the base. The deviation is 37.2°, the least this prism can give.*

The refractive index of the prism. At minimum deviation $r_1 = r_2 = r$, so from $r_1 + r_2 = A$,

$$r = \frac{A}{2} \tag{16.3}$$

and with $i = e$ and $\delta = \delta_m$, from $\delta = i + e - A$,

$$\delta_m + A = 2i \;\Rightarrow\; i = \frac{A + \delta_m}{2} \tag{16.4}$$

Snell's law at the first face, $\mu = \dfrac{\sin i}{\sin r}$, then gives

IN SHORT

Key ideas of Unit 16

- A **prism** has two plane refracting faces meeting at the **refracting angle** A. In air a glass prism always turns light, overall, towards its base.
- For any ray: $r_1 + r_2 = A$ and $\delta = i + e - A$, with $\sin i = \mu \sin r_1$ and $\sin e = \mu \sin r_2$.
- As i grows, δ falls to a **minimum deviation** δ_m and rises again. At δ_m the ray passes symmetrically: $i = e$, $r_1 = r_2 = A/2$, and the ray inside runs parallel to the base.
- **Refractive index from minimum deviation:** $\mu = \dfrac{\sin \frac{A+\delta_m}{2}}{\sin \frac{A}{2}}$.
- A **thin prism** deviates rays falling nearly normally by $\delta = (\mu - 1)A$, independent of the angle of incidence.

CAREFUL

"Small" has limits

The formula needs both small A and small i. Even a thin prism deviates more if light falls on it steeply: the 4° prism of Example 4 gives 2.0° at $i = 0$ but about 2.9° at $i = 40^\circ$. In the widget above, a 60° prism is far from thin: its deviation changes by 20° as i changes.

prisms in **prism diopters**: one prism diopter turns light by 1 cm at a distance of 1 m (an angle of about 0.57°). From $\delta = (\mu - 1)A$, a glass prism of about 1.15° gives 1 prism diopter.

- **Steering a beam.** Two thin prisms one behind the other, each turned about the beam's line (a **Risley pair**), can point a laser beam in any direction within a small cone, with no moving mirrors. They are used to aim laser beams in scanners, laser cutting and some satellite links.
- **Shaping a beam.** The light from a laser diode spreads out more in one direction than the other, giving an oval spot. Two prisms in a row widen the beam in one direction only and make the spot round (an **anamorphic prism pair**).
- **Alignment.** A thin wedge shifts an image by a small, fixed, known angle, so it is used to line up the parts of an optical system and to adjust the path of a beam a little.
- **Turning light without colors.** A thin prism of crown glass and a thinner, opposite one of flint glass can be chosen so that their color spreads cancel while a deviation remains: an **achromatic prism**, the same idea as the achromatic lens (Unit 18).

DID YOU KNOW?

Totally reflecting prisms are not mirrors

Binoculars and periscopes use right-angled prisms, but they work by total internal reflection (Unit 15, totally reflecting prisms), not by deviation through a thin prism. Compared with a silvered mirror they have three advantages: total internal reflection reflects practically all the light, while silver or aluminium reflects about 90–95% of it; there is no silvering to tarnish or scratch; and a back-silvered glass mirror gives a faint second image from its front surface, which a prism doesn't. Folding the light back and forth inside the glass also makes binoculars and periscopes shorter. A prism still reverses the picture at each reflection, just as a mirror does: that is why binoculars use an even number of reflections.

Check 4

A thin prism of angle 6° deviates a ray by 3.6°. What is its refractive index?

a 1.5 b 1.6
c 1.7 d 0.6

$$\mu = \frac{\sin\left(\frac{A + \delta_m}{2}\right)}{\sin\frac{A}{2}} \tag{16.5}$$

This is how the refractive index of glass is measured most accurately. A and δ_m are both angles, which a spectrometer measures to a fraction of a minute of arc, and finding the minimum is easy, because there the deviation hardly changes as the prism turns: you rock the prism until the image stops and turns back.

DID YOU KNOW?

Why the image "stops"

Near the bottom of the curve the deviation changes very little with i. For the prism in Figure 16.7, turning the angle of incidence 3° either side of 48.6° changes the deviation by only about 0.14°. So a spectrometer's user watching the image through a telescope sees it move, slow down, stop and move back: the stopping point is δ_m, found to a fraction of a degree even by hand.

CAREFUL

Measuring from a calculation

In equation (16.5) the angles are halved before the sine is taken. A common slip is to write $\sin(A + \delta_m)/\sin A$: for $A = 60^\circ$, $\delta_m = 37.2^\circ$ that gives 1.15 instead of 1.50.

With calculus. The same result follows from $\delta = i + e - A$. At the lowest point $d\delta/di = 0$, so $de/di = -1$. Differentiating Snell's law at both faces, with $dr_1 = -dr_2$ (since $r_1 + r_2 = A$), gives

$$\frac{de}{di} = -\frac{\cos i \, \cos r_2}{\cos r_1 \, \cos e}$$

This equals -1 when $i = e$ and $r_1 = r_2$, which is the symmetrical path.

WORKED EXAMPLE

Example 2: Finding the angle of a prism

A glass prism of refractive index 1.5 gives a minimum deviation of 40°. Find the angle of the prism.

Answer

From $\mu = \dfrac{\sin\frac{A+\delta_m}{2}}{\sin\frac{A}{2}}$ with $\delta_m = 40^\circ$:

$$1.5 = \frac{\sin\left(\frac{A}{2} + 20^\circ\right)}{\sin\frac{A}{2}} = \frac{\sin\frac{A}{2}\cos 20^\circ + \cos\frac{A}{2}\sin 20^\circ}{\sin\frac{A}{2}} = \cos 20^\circ + \cot\frac{A}{2}\sin 20^\circ$$

$$\cot\frac{A}{2} = \frac{1.5 - \cos 20^\circ}{\sin 20^\circ} = \frac{1.5 - 0.940}{0.342} = 1.638$$

$$\tan\frac{A}{2} = \frac{1}{1.638} = 0.610 \;\Rightarrow\; \frac{A}{2} = 31.4^\circ$$

The angle of the prism is **62.8°**.

WORKED EXAMPLE

Example 3: A prism for a given angle of incidence

What must the refracting angle of a prism of refractive index 1.5 be, for a ray falling on it at 50° to pass at minimum deviation?

Answer

At the first face:

$$\sin r_1 = \frac{\sin 50^\circ}{1.5} = \frac{0.766}{1.5} = 0.5107 \;\Rightarrow\; r_1 = 30.71^\circ$$

At minimum deviation $r_1 = r_2 = r$, so

$$A = 2r = 2 \times 30.71^\circ = 61.42^\circ$$

The refracting angle is **61.4°**.

YOUR TURN

Try it

1. Does the angle of minimum deviation change if the prism is put in water?
2. Find the refractive index of a prism whose angle is 60° and angle of minimum deviation 50°.

Answer

1. Yes: it becomes smaller. In water the light is bent according to the refractive index of glass relative to water, $1.5/1.33 = 1.13$, much less than 1.5, so each face bends the light less. For a 60° prism of glass, δ_m falls from 37.2° in air to about 8.7° in water: from $\sin\frac{60^\circ+\delta_m}{2} = 1.13\sin 30^\circ = 0.564$, $\frac{60^\circ+\delta_m}{2} = 34.33^\circ$.
2. $\mu = \dfrac{\sin\frac{60^\circ+50^\circ}{2}}{\sin 30^\circ} = \dfrac{\sin 55^\circ}{0.5} = \dfrac{0.819}{0.5} = \mathbf{1.638}$.

Check 3

At the minimum deviation of a prism, which of these is true?

a The angle of emergence is 90°.

b The angle of incidence equals the angle of emergence.

c The angle of incidence equals the angle of the prism.

d The deviation is zero.

16.4 Deviation produced by a small-angled prism

Take a thin prism, of small refracting angle A (a few degrees), and light falling on it nearly along the normal, so that i is small too (Figure 16.8). Then all the angles, i, r_1, r_2 and e, are small.

For a small angle θ in radians, $\sin\theta \approx \theta$. Snell's law at the first face, $\mu = \dfrac{\sin i}{\sin r_1}$, becomes

$$\mu = \frac{i}{r_1} \;\Rightarrow\; i = \mu r_1 \tag{16.6}$$

and at the second face, in the same way,

$$e = \mu r_2 \tag{16.7}$$

Putting these into $\delta = i + e - A$:

$$\delta = \mu r_1 + \mu r_2 - A = \mu(r_1 + r_2) - A = \mu A - A$$

$$\delta = (\mu - 1)A \tag{16.8}$$

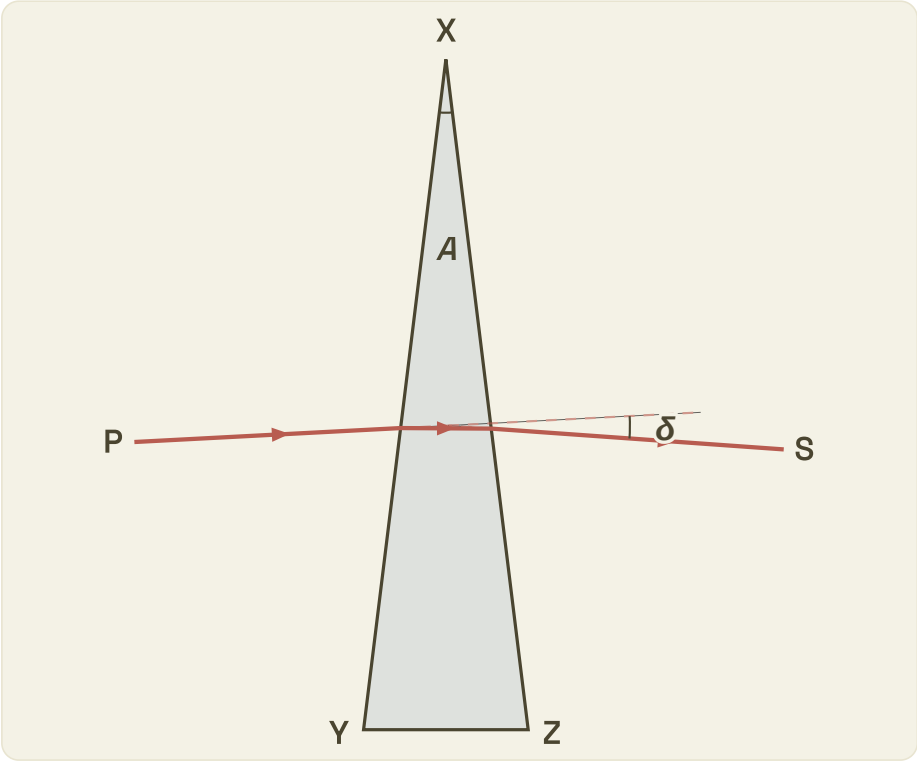


Figure 16.8 *A thin prism (here $A = 14^\circ$, $\mu = 1.5$) and a ray falling nearly along the normal: $i = 10^\circ$. All the angles are small, and the deviation, 7.1°, is very nearly $(\mu - 1)A = 7^\circ$.*

A thin prism deviates every ray that falls on it nearly normally **by the same angle**, whatever its exact angle of incidence. The deviation depends only on the refractive index and the angle of the prism. For glass ($\mu = 1.5$) a thin prism turns light through half its own angle. This is what makes a thin prism useful: like a lens of no thickness, it acts on the whole beam alike.

WORKED EXAMPLE

Example 4: A thin prism

A thin glass prism of refractive index 1.5 has a refracting angle of 4°. Through what angle does it turn a ray? What would be its angle for a deviation of 3°?

Answer

$$\delta = (\mu - 1)A = (1.5 - 1) \times 4^\circ = 2^\circ$$

It turns the ray through **2°**. For $\delta = 3^\circ$: $A = \dfrac{\delta}{\mu - 1} = \dfrac{3^\circ}{0.5} = \mathbf{6°}$.

What small-angled prisms are used for

- **Prism glasses.** An eye that turns slightly inwards or outwards sends a double image. A thin prism in the spectacle lens turns the light so that both eyes see the object in the same place. Eye doctors measure such