Unit 8

Elasticity

This unit online, with its interactives:
science-playground.com/s/11f8

Why is there no mountain on Earth twice as tall as Sagarmatha (Everest)? Not because the Himalaya stopped rising: the plates are still pushing. The answer is in the rock at the bottom. Pile up enough rock and the rock underneath can no longer spring back: it starts to flow and crumble, and the mountain sinks and spreads. A pencil, a rough number for rock and one line of working put the limit at about 10 km. Sagarmatha, at 8,849 m, is close to it.

That is what this unit is about: how solids push back when we pull, squeeze or twist them, how far they can be pushed before they give way, and how much energy they store on the way. It is the physics behind every spring, every steel cable on a Nepali suspension bridge, every rubber band and every building that must ride out an earthquake.

8.1 Introduction

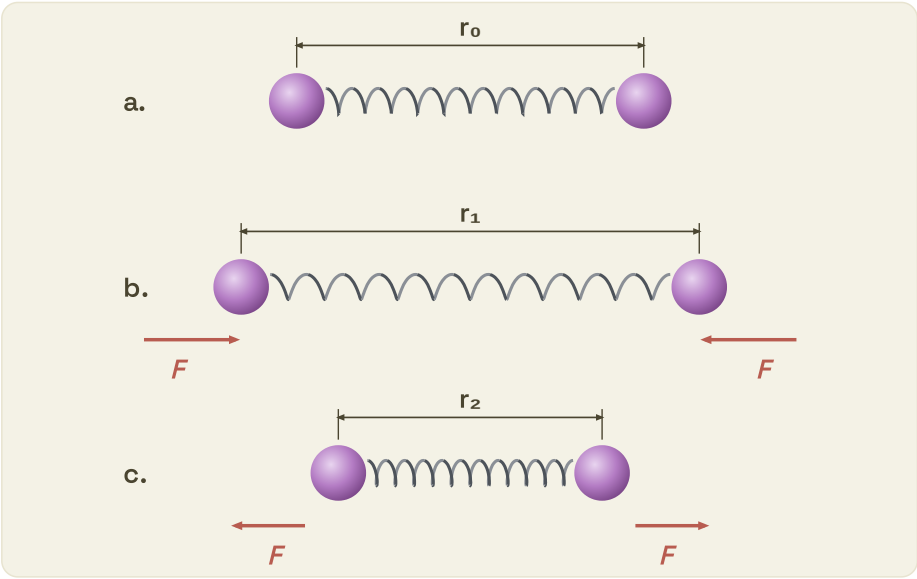


Figure 8.1 *a. The spring at its natural length r_0: no force. b. Stretched to r_1: the restoring forces F pull the marbles back together. c. Squeezed to r_2: the restoring forces push them apart. In both cases the restoring force points back towards the natural length.*

The force you apply in either case changes the size or shape of the body: it is a **deforming force**. While the body is deformed, a force sets up inside it that tries to bring it back to its original shape: the **restoring force**. When you remove the deforming force, the restoring force returns the body to its original size and shape.

Elasticity is the property of matter by which a body regains its original shape and size when the deforming force is removed.

Pull the spring hard enough and it doesn't come back: it stays longer, with its coils pulled open. It has been stretched beyond its **elastic limit**, the greatest deformation (or deforming force) from which it can still recover

completely. Beyond it the body is permanently deformed: it is in the **plastic** stage (engineers say *damaged*, or *yielded*). **Plasticity** is the property of a body by which it keeps its new shape after the deforming force is removed.

CAREFUL

"More elastic" means harder to deform

In everyday speech rubber is "elastic" because it stretches easily. In physics, a material is **more elastic** if it takes a **larger** force to deform it by a given amount, and it still springs back. A steel wire needs thousands of times more force than a rubber cord of the same size to stretch by the same amount, so **steel is more elastic than rubber**; iron is more elastic than rubber too. In the same sense solids are more elastic than liquids, and liquids far more elastic than gases (you can squash the air in a syringe with your thumb, but not the water).

Where elasticity comes from

Every solid is made of atoms held near one another by electric forces. Two neighboring atoms sit at the separation r_0 where the attraction between them and their repulsion balance: the **equilibrium separation**, about 0.1 to 0.3 nm.

- Pull them apart a little (separation more than r_0), and the **attraction** wins: a force pulls them back.
- Push them together (separation less than r_0), and the **repulsion** wins: a force pushes them apart.

So each bond behaves like a tiny spring, and a solid is a three-dimensional net of such springs. Stretch the solid and you stretch billions of bonds a tiny bit each; their restoring forces add up to the restoring force you feel. This is why elasticity comes from **short-range forces between molecules**.

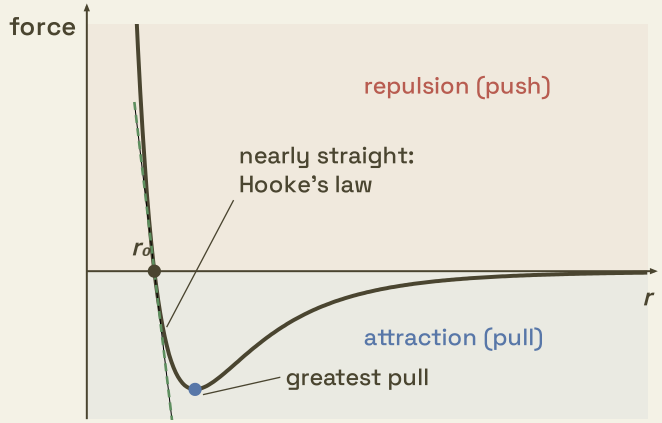


Figure 8.2 *The force between two neighboring atoms against their separation r (worked out from the Lennard-Jones model). At r_0 it is zero. Closer, it is a strong repulsion; farther, an attraction that rises to a greatest value and then dies away. Near r_0 the curve is almost a straight line through zero: a restoring force proportional to the displacement, which is Hooke's law.*

Two things follow from the curve:

- Near r_0 the curve is nearly straight, so for **small** deformations the restoring force is proportional to the deformation. That is Hooke's law (Section 8.2).
- The attraction has a greatest value. Pull harder than that and the bonds can no longer pull the atoms back. In a metal, what really happens is that whole layers of atoms **slip** past one another into new places and stay there: that is plastic deformation. Pull harder still and the solid breaks.

Numerical problems

19. Young's modulus for steel is 2.0×10^{11} N/m². Find the work done in stretching a steel wire 100 cm long, of cross-sectional area 0.030 cm², when a load of 100 N is slowly applied, without the elastic limit being reached.
20. A uniform steel wire of density 7800 kg/m³ has a mass of 16 g and is 2.50 m long. It stretches by 1.2 mm when pulled by a force of 80 N. Find a. the Young's modulus of the steel and b. the energy stored in the wire.
21. The rubber cord of a catapult has a cross-sectional area of 1.0 mm² and an unstretched length of 10.0 cm. It is stretched to 12.0 cm and then released to throw a stone of mass 5.0 g. From energy considerations, or otherwise, find the speed with which the stone is thrown. (Y for rubber = 5.0×10^{8} N/m²)
22. A copper wire 1 m long and a steel wire 2 m long, of the same cross-sectional area, are joined end to end. A force applied to the combination stretches its total length by 1 cm. Find how much each wire stretches. (Y for copper = 1.2×10^{11} N/m², for steel = 2.0×10^{11} N/m²)
23. A uniform wire of density 8000 kg/m³ has a mass of 20 g and is 2.5 m long. It stretches by 1 mm when pulled by a force of 80 N. Find the Young's modulus of the metal and the energy stored in the wire.
24. A wire 3.0 m long, of cross-sectional area 1.0×10^{-6} m², has a mass of 15 kg hung on it. What is the stress in the wire? If the Young's modulus of its material is 2.0×10^{11} N/m², what is the extension x? How much energy is stored in the stretched wire? (g = 9.8 m/s²)
25. A steel wire 2.00 m long must stretch by no more than 0.25 cm when a tensile force of 400 N is applied to each end. What is the least diameter it can have? (Y = 20×10^{10} N/m²)
26. A vertical brass rod of circular cross-section is loaded by placing a 5 kg weight on top of it. The rod is 50 cm long, its radius is 1 cm, and the Young's modulus of brass is 3.5×10^{10} N/m². Find a. the contraction of the rod and b. the energy stored in it. (g = 9.8 m/s²)
27. In a materials testing laboratory, a wire made of a new alloy breaks when a tensile force of 90.8 N is applied to each end. If the wire's diameter is 1.84 mm, what is the breaking stress of the alloy?
28. A metal rod 4.00 m long, of cross-sectional area 0.50 cm², stretches by 0.20 cm under a tension of 5000 N. What is the Young's modulus of the metal?
29. A sample of oil of initial volume 600 cm³ is put under an extra pressure of 3.6×10^{6} Pa, and its volume decreases by 0.45 cm³. What is the bulk modulus of the oil? Its compressibility?

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

Check 1

A copper wire is stretched beyond its elastic limit and the load is then removed. The wire:

a returns to its original length
b stays a little longer than it was
c becomes shorter than it was
d breaks at once

Elasticity in use

- **Springs are made of steel, not copper.** A spring balance must give the same reading for the same load every time, so its spring must spring back exactly and must not take a permanent set. Steel is more elastic than copper (it needs a greater force for the same extension) and has a much higher elastic limit, so a steel spring stays true far longer.
- **Steel rods in reinforced concrete (RCC).** Concrete is strong when squeezed but cracks easily when pulled; the steel bars cast inside it carry the pulling forces. In an earthquake a building sways back and forth. Within their elastic limit the bars spring back each time; and if a shock is so strong that they go past it, good steel bars bend and stretch (they are **ductile**, Section 8.7) instead of snapping, so the building holds together long enough for people to get out. That is why Nepal's building code asks for proper steel bars, bent hooks and closely spaced rings (ties) round them.
- **Twisted (stranded) cables.** High-tension power lines and crane ropes are made of many thin wires twisted together, not one thick rod. A thin wire bends with little strain, so the cable is flexible and doesn't kink; if one strand cracks, the others still hold; and thin strands suffer less from metal fatigue (Section 8.8) than a solid rod.

A STORY FROM SCIENCE

Hooke, who has no face

Robert Hooke was born on 18 July 1635 at Freshwater, on the Isle of Wight in England. As a young man he worked for Robert Boyle, and around 1658 he built Boyle's famous air pump, which Boyle used to discover his law of gases. In 1662 he became the Royal Society's **curator of experiments**, the man who had to show the Fellows a new experiment at every meeting, and in 1663 a Fellow himself. He looked at fleas, cork and snowflakes through a microscope and published the drawings in *Micrographia* (1665), where he named the little boxes in cork **cells**.

Hooke said he had found his law of springs around 1660, but he was secretive. In 1676 he printed it only as an anagram, *ceiiinosssttuv*, so that he could prove later that he had known it first. Two years later he revealed the answer: *Ut tensio, sic vis*, Latin for "as the extension, so the force". He used springs to make the balance spring that keeps a watch ticking evenly.

Hooke quarreled with Isaac Newton for years, and after Hooke died in 1703, no portrait of him survived. Pictures labeled "Robert Hooke" in many books and websites are of other men. The painting below was made in 2004 by the artist Rita Greer, from descriptions written by people who knew him.



Figure 8.3 *Robert Hooke (1635–1703) as the artist Rita Greer imagined him in 2004, from his friends' descriptions: no true portrait of him survives. She put a coiled spring on his desk.*
Photo: Rita Greer, FAL.

8.2 Hooke's law

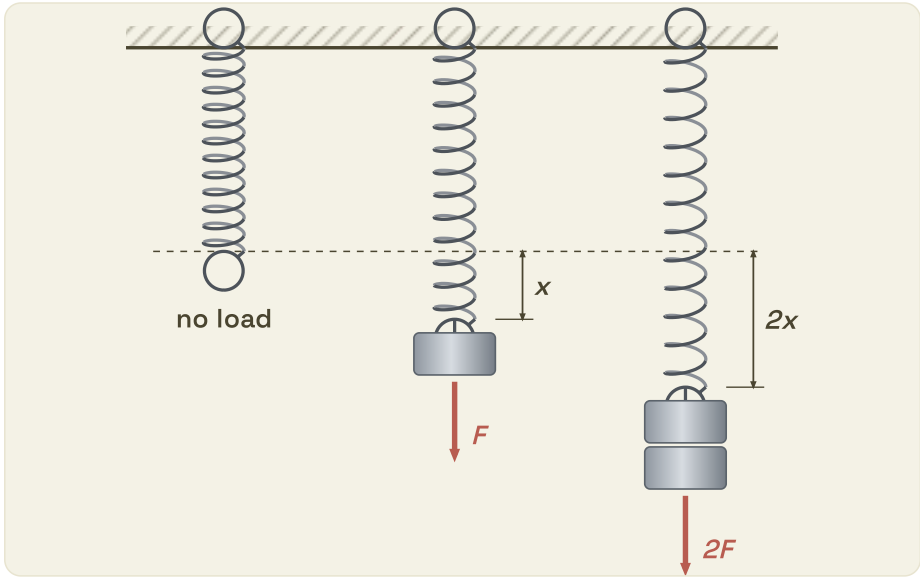


Figure 8.4 *A spring hung from a support. A load that pulls with force F stretches it by x; twice the force, 2F, stretches it twice as far, 2x. The spring pulls back on the load with $F = -kx$.*

Hang a spring from a hook and add loads one at a time, measuring the extension each time. Within the elastic limit you find that equal loads give equal extensions: the **extension is directly proportional to the load**. This is **Hooke's law**:

> Within the elastic limit, the extension of a body is directly proportional to the force (load) applied to it.

$$x \propto F \quad \Rightarrow \quad F = kx$$

The spring pulls back on whatever stretches it with a force of the same size, towards its natural length, opposite to the extension, so the **restoring force** is

$$F = -kx \tag{8.1}$$

Check 18

10. Shearing strain is measured by the:

a angle of shear
b angle of twist
c decrease in volume
d increase in volume

Check 19

11. A ball's volume decreases by 0.01% under a pressure of 100 atmospheres. The bulk modulus of its material is about:

a 1×10^{10} N/m²
b 2×10^{11} N/m²
c 1×10^{13} N/m²
d 1×10^{11} N/m²

Check 20

12. The delay with which an elastic body regains its original shape after the deforming force is removed is called:

a elastic fatigue
b elasticity
c plasticity
d elastic after-effect

15.

a. What is meant by Poisson's ratio?
b. Define elastic after-effect.
c. Find the increase in energy of a brass bar 0.5 m long, of cross-sectional area 1×10^{-4} m², when it is compressed along its length by a load of 5 kg. (Y for brass = 1.0×10^{11} N/m², g = 10 m/s²)

16.

a. What is elastic potential energy? Derive an expression for the energy stored in a stretched wire.
b. A uniform rod of a material with Poisson's ratio 0.2 is given a longitudinal strain of 4×10^{-3}. Find the percentage change in its volume.
c. A copper wire of cross-sectional area 0.01 cm² is under a tension of 25 N. Find the decrease in its cross-sectional area. (Y for copper = 1.1×10^{11} N/m², $\sigma = 0.32$)

17.

a. Discuss the different moduli of elasticity.
b. What force would be needed to stretch a steel wire of cross-sectional area 1×10^{-4} m² to double its length, if Hooke's law held that far? (Y for steel = 2.0×10^{11} N/m²)
c. A steel wire and a copper wire of equal cross-sectional area are joined end to end. When the combination is under tension, the two wires stretch by equal amounts. Find the ratio of the length of the steel wire to that of the copper wire. (Y for steel = 2.0×10^{11} N/m², for copper 1.1×10^{11} N/m²)

18.

a. What are the elastic limit and the breaking stress?
b. Define elastic hysteresis.
c. A sample of fishing line is 1.0 m long, with a circular cross-section of radius 0.25 mm. When a weight is hung on it, it stretches by 50 mm and the stress is 2.0×10^{8} Pa. Find i. its cross-sectional area, ii. the weight hung on it, iii. the strain, and iv. its Young's modulus.

Check 10

2. A light steel wire of force constant k hangs from a ceiling. A mass m is attached to its lower end while held at the wire's natural length, and then suddenly released. The greatest elongation of the wire is:

a mg/k
b 2mg/k
c mg/2k
d 3mg/k

Check 11

3. Which is the correct expression for Poisson's ratio (R the radius and L the length of a wire)?

a $\frac{\Delta R}{L} \times \frac{R}{\Delta L}$
b $\frac{\Delta R}{R} \times \frac{L}{\Delta L}$
c $\frac{\Delta R}{\Delta R} \times \frac{L}{\Delta L}$
d $\frac{\Delta R}{R} \times \Delta L \times L$

Check 12

4. A rod is fixed between two rigid supports and then heated. The stress set up in the rod is:

a compressive stress
b tensile stress
c shear stress
d no stress

Check 13

5. Which of these has the highest elasticity?

a rubber
b sponge
c copper
d steel

Check 14

6. How does Young's modulus change when the temperature rises?

a it increases
b it decreases
c it remains unchanged
d it changes erratically

Check 15

7. Substances that can be stretched to large strains and still spring back are called:

a brittle
b ductile
c plastic
d elastomers

Check 16

8. A shearing stress applied to a body changes its:

a length
b breadth
c shape
d volume

Check 17

9. A uniform wire of mass m, length L, cross-sectional area A and Young's modulus Y hangs from a ceiling. Its elongation under its own weight is:

a zero
b mgL/2AY
c 2mgL/AY
d mgL/AY

Answer these questions

13. a. State Hooke's law.

b. Does Hooke's law apply to all materials?

The constant k is the **force constant** (or **spring constant**, or **stiffness**): the force needed per unit extension. Its unit is the newton per meter, N/m. A stiff spring has a large k; a soft one, a small k.

WORKED EXAMPLE

A spring's force constant

A spring stretches by 4.0 cm when a 2.0 kg mass hangs from it. Find its force constant, and how far it stretches with 3.0 kg. ($g = 9.8$ m/s²)

Answer

Load: $F = mg = 2.0 \times 9.8 = 19.6$ N; extension $x = 0.040$ m.

$$k = \frac{F}{x} = \frac{19.6}{0.040} = 490 \text{ N/m}$$

With 3.0 kg: $x = \frac{F}{k} = \frac{3.0 \times 9.8}{490} = 0.060$ m = **6.0 cm**. (Or simply: 1.5 times the load, 1.5 times the extension.)

From a spring to a wire: stress and strain

A wire obeys the same law, but its force constant depends on its size: a **longer** wire stretches more under the same load (every part of it stretches, and the stretches add up), and a **thicker** wire stretches less (the load is shared by more bonds). For a wire of length L and cross-sectional area A, k turns out to be proportional to $\frac{A}{L}$, and it helps to take the size out:

$$F = kx \quad \Rightarrow \quad \frac{F}{A} = \left(\frac{kL}{A}\right)\frac{x}{L}$$

Here $\frac{F}{A}$, the force per unit area, is the **stress**, and $\frac{x}{L}$, the extension per unit length, is the **strain**. For a given material $\frac{kL}{A}$ is the same whatever the wire's size, so **stress ∝ strain**:

$$\frac{\text{stress}}{\text{strain}} = E \tag{8.2}$$

E is the **modulus of elasticity** of the material. So Hooke's law can also be stated:

> Within the elastic limit, stress is directly proportional to strain.

The first form is about one particular spring or wire; this second form is about the **material**, whatever its shape. Compare: k for a wire is $\frac{EA}{L}$.

Check 2

Two springs stretch 2 cm and 5 cm under the same load. Which has the larger force constant?

a the one that stretches 2 cm
b the one that stretches 5 cm
c they are equal, since the load is the same
d it depends on how long they are

Experimental verification

Two long, identical wires are hung side by side from the same rigid support (Figure 8.5):

- the **reference wire** A carries the **main scale** M and a small constant load that keeps it straight;
- the **experimental (test) wire** B carries a **vernier scale** V, just beside M, and a hanger for loads.

Why two wires? If the support sags a little under the load, or the room warms up and both wires expand, both scales move together and the reading doesn't change. The vernier measures only how much B stretches more than A, which is the extension caused by the load.

1. With only a small starting load on B (to take the kinks out), note the vernier reading.
2. Add equal loads (say 0.5 kg or 1 kg) one at a time, waiting a minute after each, and note the reading each time. Stay well within the elastic limit.

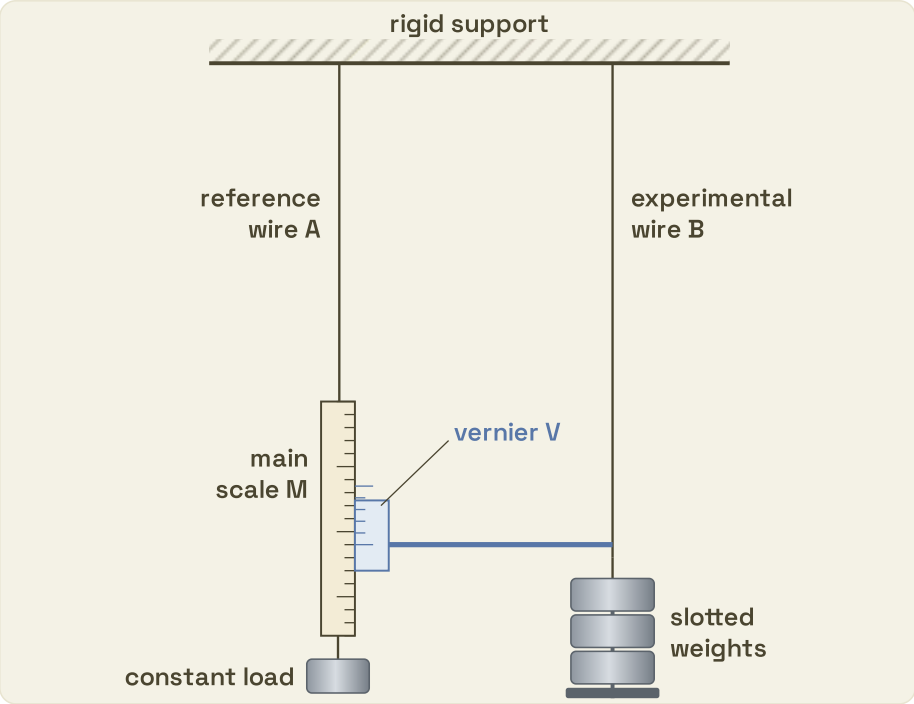


Figure 8.5 *Verifying Hooke's law. Two identical wires hang from the same rigid support. The reference wire A carries a main scale and a small constant load that keeps it straight; the experimental wire B carries the vernier and the hanger for the loads. The vernier reads how much B stretches relative to A.*

3. Then take the loads off one at a time, noting the readings again (they should match the first set, showing the wire has not taken a permanent set).
4. For each load, the extension is the reading minus the starting reading; take the mean of the loading and unloading values.

Plotting load against extension gives a **straight line through the origin**: the extension is proportional to the load, $x \propto F$, so $F = kx$ and, dividing by A and L as above, stress ∝ strain. Hooke's law is verified. (The same apparatus measures the Young's modulus of the wire, Section 8.4.)

8.3 Stress, strain, elasticity and plasticity

Stress

When a body is deformed, the restoring force inside it acts across every section of it. The **restoring force per unit area** is the **stress**. When the body is in equilibrium, the restoring force equals the applied force, so in practice

$$\text{stress} = \frac{\text{force}}{\text{area}} = \frac{F}{A}$$

Its SI unit is the newton per square meter, N/m², also called the **pascal** (Pa); its dimensions are $[ML^{-1}T^{-2}]$, the same as pressure.

Stress and pressure are both a force per unit area, but they are not the same thing:

sharp corner or a flaw, and grow a little with each cycle. This loss of strength under repeated stress is **elastic fatigue** (or **fatigue failure**), a little like a person who tires after doing the same work over and over.

Its effects:

- A wire can be broken by repeated bending even though the stress never reaches its breaking stress; it can even be cut into pieces by hand, with no tools, just by bending it back and forth at the same spot.
- After many years of service, railway tracks, bridges and aircraft are inspected for fatigue cracks, and replaced or declared unsafe.
- After long use, a spring balance's spring gives wrong readings, because the spring has lost some of its elasticity.

A STORY FROM SCIENCE

The airliner that taught engineers about fatigue

In 1954 two de Havilland Comets, the world's first jet airliners, broke apart in the sky within a few months. To find out why, engineers put a whole Comet in a water tank and pumped its cabin up and down, as each flight's climb and descent does, again and again. After the equivalent of about 3,000 flights (some 1,800 of them in the tank) the cabin split open, from a fatigue crack at the corner of a window; wreckage raised from the sea showed a crack of the same kind at the corner of an opening in the roof. Since then, the corners of aircraft windows and openings are shaped and strengthened to keep the stress there low, every part is designed for its fatigue life, and aircraft are inspected for cracks on a fixed schedule.

IN SHORT

Key ideas of Unit 8

- **Elasticity**: a body regains its shape and size when the deforming force is removed; **plasticity**: it keeps the new shape. The **elastic limit** separates the two. In physics, "more elastic" means harder to deform: steel is more elastic than rubber.
- **Hooke's law**: within the elastic limit, F = −kx (force constant k in N/m), or stress ∝ strain.
- **Stress** = F/A (N/m², Pa): tensile, compressive (normal), or tangential (shearing). **Strain** = change ÷ original (no unit): longitudinal x/L, volumetric −ΔV/V, shear φ ≈ x/L.
- **Moduli**: Young's Y = FL/(Ax) (length); bulk B = P/(−ΔV/V) (volume; compressibility = 1/B); rigidity η = (F/A)/φ (shape). Y from experiment = slope × L/A.
- **Elastic PE** = ½Fx = ½kx² = ½(YA/L)x²; **energy density** = ½ stress × strain = ½Y(strain)².
- **Poisson's ratio** σ = lateral strain ÷ longitudinal strain, between 0 and 0.5 for ordinary materials; ΔV/V = ε(1 − 2σ).
- **Stress–strain curve**: proportional limit, elastic limit, yield, strain hardening, ultimate stress, necking, fracture. Ductile materials have a long plastic region, brittle ones almost none; elastomers (rubber) stretch far without obeying Hooke's law.
- **Elastic after-effect** (a delay in recovering), **hysteresis** (the loop's area is heat each cycle), **fatigue** (breaking under repeated loads below the breaking stress).

Exercise

Choose the correct option

Check 9

1. Which modulus of elasticity do solids, liquids and gases all have?

a shear modulus
b bulk modulus
c Young's modulus
d all of the above

- For **quartz** and **phosphor bronze** the after-effect is negligible; for glass fiber it is considerable. That is why the suspension fibers of moving-coil galvanometers and other sensitive instruments are made of quartz or phosphor bronze: after each deflection they must return to zero at once.
- A perfectly elastic body would have **zero** after-effect; a perfectly plastic (inelastic) body never returns at all, so its after-effect is, in effect, **infinite**.

Elastic hysteresis

Stretch a rubber band and let it go back, plotting the stress against the strain all the way. The graph on the way back (unloading) does **not** follow the graph on the way out (loading): for the same strain, the stress is lower while unloading. The band still returns to its original length, but along a different path. This lagging of the strain behind the stress is **elastic hysteresis**, and the closed figure the two curves make is the **hysteresis loop**.

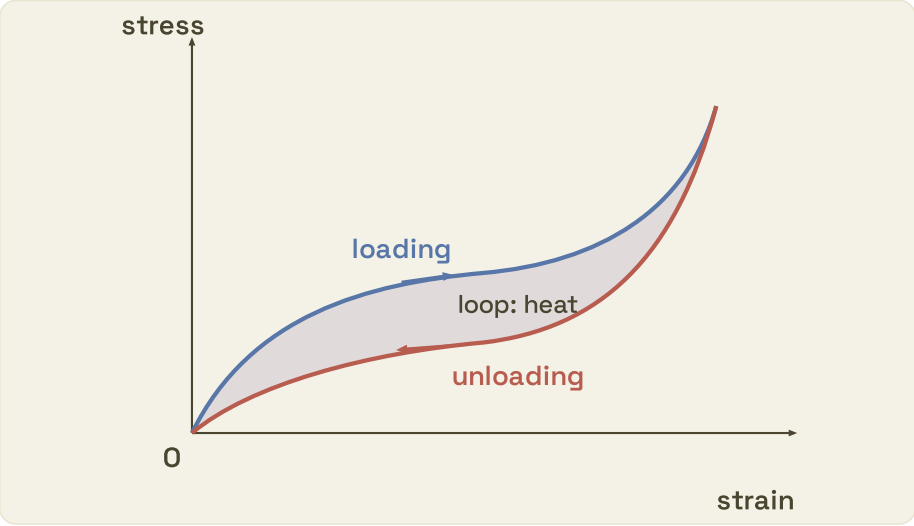


Figure 8.17 *Elastic hysteresis in rubber. The loading curve (going out) lies above the unloading curve (coming back). The area under the loading curve is the work done stretching it; the area under the unloading curve is the work it gives back. The shaded loop between them is the energy turned into heat in one cycle.*

The area under the loading curve is the work done on the rubber per unit volume; the area under the unloading curve is the work it gives back. The difference, the **area of the loop**, is energy that has not been stored but turned into **heat** inside the rubber, every cycle. So:

- **Rubber absorbs vibration.** Rubber mounts under engines and machines, and rubber pads (bearings) under bridges and some buildings, take in the energy of each shake and turn it into heat, because rubber has a large hysteresis loop.
- **Tyres get warm.** A tyre is squashed and released at every turn of the wheel; its hysteresis loss heats it, and is part of the rolling resistance that costs fuel.
- For a car's **suspension spring**, or a watch's balance spring, you want the opposite: steel, whose loop is so thin that almost all the energy comes back.

Elastic fatigue

Bend a paper clip back and forth: after a dozen or so bends it snaps, though no single bend was strong enough to break it. A material that is loaded and unloaded over and over, many thousands or millions of times, can crack and break at a stress **well below** its breaking stress, or even below its elastic limit. Tiny cracks start at a scratch, a

- **Pressure** is the push of a fluid (a liquid or a gas) on a surface. At any point in a fluid at rest it is the same in every direction, and it is always a push, at right angles to the surface. It is a **scalar**.
- **Stress** is the force per unit area **inside a solid**, set up by stretching, squeezing, shearing, twisting or bending it. It can pull as well as push, it can act along a surface as well as across it, and at one point its size depends on which way the surface faces.

> CAREFUL
> **Not a vector**
>
> Some books call stress and pressure vectors. Pressure has no direction of its own: it is a scalar. Stress isn't a simple vector either (to describe it fully at a point needs a table of nine numbers, a *tensor*, beyond Grade 11). At this level, write a stress as a force per unit area and say whether it is tensile, compressive or shearing.

Types of stress

Stress is named by the direction of the force relative to the surface it acts on.

1. Normal stress: the force acts **at right angles** (normal) to the surface. It is of two kinds:

- **Tensile stress**: the forces pull outwards along the length, and the body gets **longer** (a crane's cable, a bridge's hangers, a guitar string).
- **Compressive stress**: the forces push inwards along the length, and the body gets **shorter** (a pillar, the legs of a table, your thigh bone).

When a body is squeezed from **all sides** equally, as a stone at the bottom of a lake is, the normal stress is the same everywhere: it is the pressure, and it changes the body's volume. This is the stress of the bulk modulus (Section 8.4).

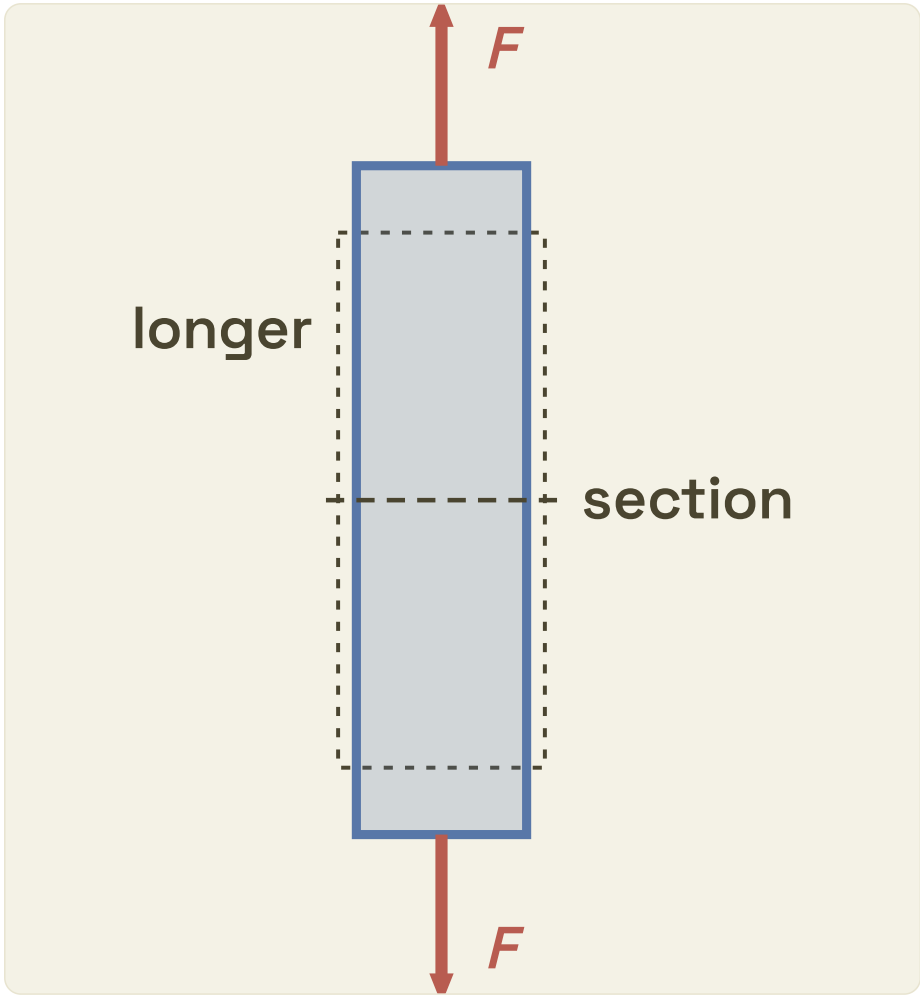


Figure 8.6 *Tensile stress: equal and opposite forces F pull along the bar, which gets a little longer (dotted: the bar before; the change is drawn much larger than real). Across any section (dashed), each part pulls the other with F.*

2. Tangential (shearing) stress: the force acts **along** (parallel to) the surface. Hold the bottom face of a thick book flat on a table and push its top cover sideways: the book leans over, its pages sliding over one another. The force per unit area of the face it acts on is the **tangential** or **shearing stress**.

In Figure 8.8 the tangential force F acts along the upper face AB while the lower face CD stays fixed. A moves to A′ and B to B′, with AA′ = BB′ = x. The shearing stress is $\frac{F}{A}$, where A is the area of the face AB (the area under shear).

Strain

A deforming force changes the body's length, volume or shape. **Strain** is the change in a dimension divided by the original value of that dimension:

CAREFUL

Elastic limit and proportional limit

The **proportional limit** (A) is where stress stops being proportional to strain; the **elastic limit** (B) is where the wire stops coming back to its original length. For most metals they are so close that many books treat them as one point. Rubber is the opposite case: it has no proportional part at all, yet springs back from a huge stretch.



Figure 8.16 *The two halves of an aluminium test bar pulled until it broke. The bar narrowed towards the break (necking, clearest on the left half) before it tore, with a rough, slanted break: signs of a ductile metal.*
Photo: Sigmund, CC BY-SA 3.0

Ductile and brittle materials

- **Ductile** materials have a long plastic region between the elastic limit and fracture: they stretch a long way, plastically, before they break. Most metals (copper, mild steel, aluminium, gold, silver) are ductile; that is what lets them be drawn into wires, bent into shape and hammered into sheets. A long plastic region means great ductility.
- **Brittle** materials break soon after (or at) the elastic limit, with almost no plastic region. Glass, cast iron, chalk, concrete and pottery are brittle: they snap instead of bending.

For bending a wire into shape, as for electric wiring, the steel bars of RCC or a jeweler's gold, ductile materials are wanted.

Elastomers are materials that can be stretched to very large strains and still spring back, but that **don't obey Hooke's law** even within their elastic limit: their stress–strain curve is not a straight line anywhere. Rubber is the best-known elastomer: a rubber band can be stretched to five or six times its length and returns. Its long, coiled molecules straighten out as it stretches and curl back when it is released.

Check 8

On the stress–strain curve of a ductile metal, the greatest stress the wire can bear is at:

a the elastic limit
b the yield point
c the ultimate stress point (D)
d the fracture point (E)

8.8 Elastic after-effect, hysteresis and fatigue

Elastic after-effect

Some bodies regain their shape as soon as the deforming force is removed; others take some time. The delay in regaining the original shape after the deforming force has been removed is the **elastic after-effect**.

8.7 The stress–strain curve

Clamp a metal wire (or a test bar) in a machine that pulls its ends apart slowly, measuring the stress and the strain all the way until it breaks. The graph of stress against strain is the material's **stress–strain curve**. For a ductile metal such as mild steel it has these parts (Figure 8.15).

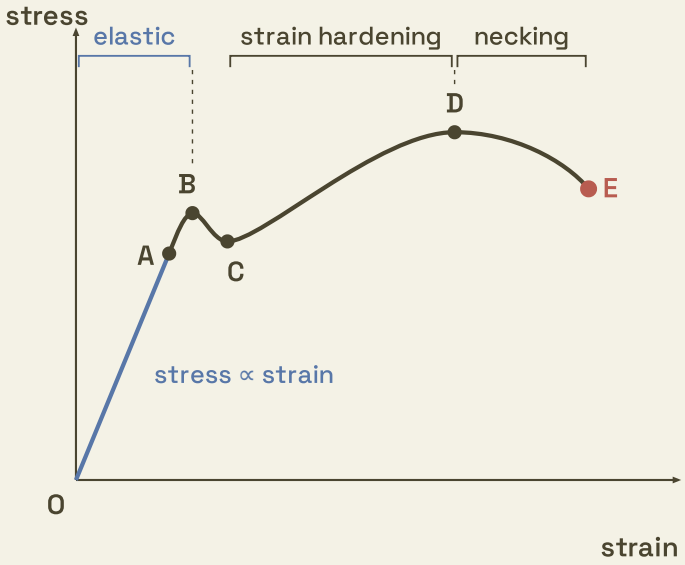


Figure 8.15 *The stress–strain curve of mild steel, as it is usually drawn: the elastic part is drawn much longer than real (in real steel it ends at a strain of about 0.1%, while the bar breaks at about 30%). O to A: stress ∝ strain. A: proportional limit. B: elastic limit, the upper yield point. C: lower yield point. C to D: plastic flow and strain hardening. D: ultimate (greatest) stress. D to E: necking. E: fracture.*

- **O to A: Hooke's law.** The graph is a straight line: stress is proportional to strain. **A** is the **proportional limit**. The slope of OA is the Young's modulus.
- **A to B: still elastic.** The line curves a little: stress is no longer proportional to strain, but if the load is removed the wire still comes back to its original length. **B** is the **elastic limit**; for mild steel it is also the **upper yield point**, and A and B are very close together.
- **B to C: yielding.** Past B the wire **yields**: it suddenly stretches without needing more stress (the stress even drops a little, to the **lower yield point** C). If the load is removed now, the wire doesn't return to its original length: it keeps a **permanent set**. Between B and C the wire is partly elastic and partly plastic, and C is often called simply the **yield point**.
- **C to D: plastic flow and strain hardening.** The wire stretches a great deal, and it takes a rising stress to stretch it further: the metal becomes harder as it is worked, which is called **strain hardening**. **D** is the greatest stress the wire can take: the **ultimate tensile stress** (or tensile strength). Some books call D the **breaking stress**, since it is the greatest stress before breaking.
- **D to E: necking and fracture.** Past D the wire starts to get thinner at one weak place, a **neck**. All further stretching happens in the neck, so a smaller and smaller load stretches it, and it breaks at **E**, the **fracture point**. (The stress here is worked out on the original area; the true stress in the shrinking neck is still rising.)

Material	Poisson's ratio
Cork	about 0
Concrete	about 0.1–0.2
Glass	about 0.2–0.25
Steel	about 0.3
Copper	about 0.34
Rubber	nearly 0.5

How the volume changes

A wire of diameter D and length L has volume $V = \frac{\pi}{4} D^2 L$. When it stretches, its length grows by the fraction $\frac{x}{L}$ (the strain ε, epsilon) and its diameter shrinks by the fraction $\sigma\varepsilon$. For small changes, the fractional change in a product is the sum of the fractional changes of its factors (D counts twice, since it is squared):

$$\frac{\Delta V}{V} = \frac{x}{L} - 2\frac{\Delta D}{D} = \varepsilon - 2\sigma\varepsilon = \varepsilon(1 - 2\sigma)$$

- If σ were exactly 0.5, the volume wouldn't change at all: rubber, with σ close to 0.5, keeps almost the same volume however it is stretched.
- For σ below 0.5 the volume grows a little when the wire is stretched. A σ above 0.5 would make a stretched wire shrink in volume, which no material does (it would mean squeezing it from all sides made it bigger), so σ can never be more than 0.5; for ordinary materials it lies between **0 and 0.5**.
- **Cork**, with σ about 0, hardly gets fatter when squeezed along its length. That is why a cork can be pushed into a bottle's neck: rubber, which bulges sideways when pushed, jams.

DID YOU KNOW?

Materials that get fatter when stretched

Some specially made foams, and a few crystals and fabrics, have a **negative** Poisson's ratio: pull them and they get **fatter**. They are called *auxetic* materials. Their inside is built of re-entrant (inward-pointing) cells that unfold when pulled. They are being used for blast-resistant panels, better-fitting medical stents and padding that hugs whatever hits it.

WORKED EXAMPLE

A steel wire gets thinner

A steel wire 2.0 m long and 1.0 mm in diameter is stretched by 1.0 mm. Poisson's ratio for steel is 0.30. By how much does its diameter decrease?

Answer

Longitudinal strain: $\frac{x}{L} = \frac{1.0 \times 10^{-3}}{2.0} = 5.0 \times 10^{-4}$.

Lateral strain: $\sigma \times$ longitudinal strain $= 0.30 \times 5.0 \times 10^{-4} = 1.5 \times 10^{-4}$.

$$\Delta D = 1.5 \times 10^{-4} \times D = 1.5 \times 10^{-4} \times 1.0 \text{ mm} = 1.5 \times 10^{-4} \text{ mm}$$

About **0.15 µm**, a few thousandths of the thickness of a human hair, far too small to see, but there.

Check 7

A wire of a material with Poisson's ratio 0.25 is stretched by a strain of 0.002. Its volume:

a stays the same
b decreases by 0.1%
c increases by 0.1%
d increases by 0.2%

YOUR TURN

Two wires of the same metal

Wires A and B are made of the same metal. A has twice the diameter and three times the length of B. Each is stretched by the same tension, within the elastic limit. Find the ratio of the energy stored in A to that in B.

Hint

With the force fixed, write the energy in terms of F: $E = \frac{1}{2}Fx = \frac{F^2 L}{2AY}$.

Answer

$E = \frac{F^2 L}{2AY}$, with F and Y the same for both, so $E \propto \frac{L}{A} \propto \frac{L}{d^2}$.

$$\frac{E_A}{E_B} = \frac{L_A}{L_B} \times \frac{d_B^2}{d_A^2} = 3 \times \frac{1}{4} = \frac{3}{4}$$

The ratio is **3 : 4**.

8.6 Poisson's ratio

Stretch a rubber band and it gets thinner as well as longer; squash a ball of dough and it bulges out sideways. Almost every material does the same, only far less visibly: a wire that is stretched becomes slightly **thinner**, and a rod that is compressed becomes slightly **fatter**.

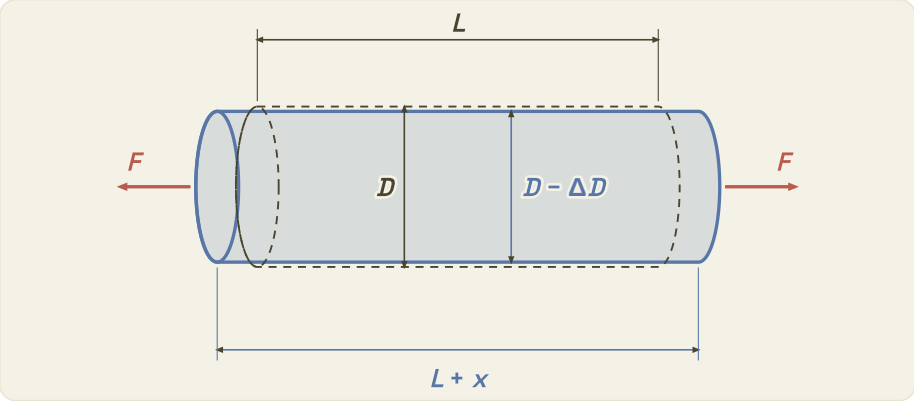


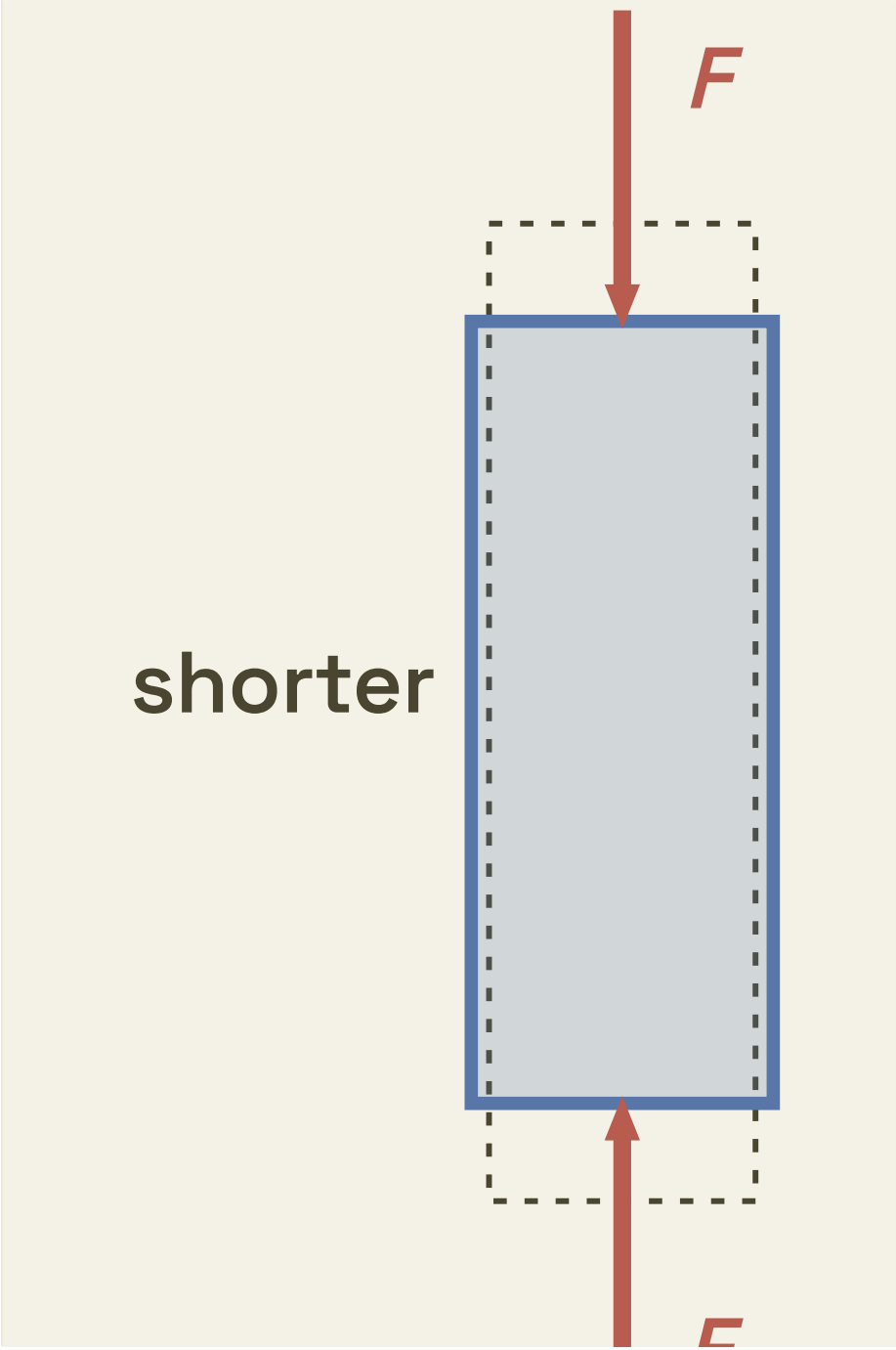
Figure 8.14 *A rod of length L and diameter D, pulled by forces F at its ends, stretches to L + x and becomes thinner, D − ΔD (the changes are drawn much larger than real).*

- The change in length per unit length, $\frac{x}{L}$, along the force, is the **longitudinal strain**.
- The change in diameter per unit diameter, $\frac{\Delta D}{D}$, at right angles to the force, is the **lateral strain**.

Within the elastic limit, the lateral strain is proportional to the longitudinal strain. The ratio of the lateral strain to the longitudinal strain is **Poisson's ratio**, σ (sigma):

$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{\Delta D / D}{x / L} = \frac{\Delta D \cdot L}{D \cdot x} \tag{8.7}$$

where ΔD is the **decrease** in diameter when the length increases by x. (Written with signs, the lateral strain is −ΔD/D, opposite in sign to x/L, and σ = −(lateral strain)/(longitudinal strain), which comes to the same positive number.) It is a ratio of two strains, so it has no unit.



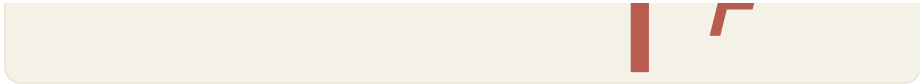


Figure 8.7 *Compressive stress: the forces push along the bar, which gets a little shorter (dotted: the bar before).*

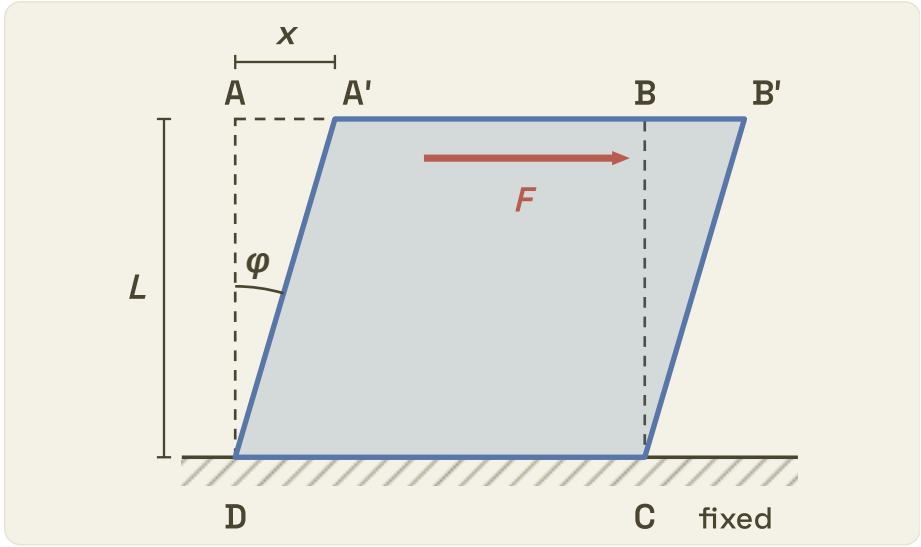


Figure 8.8 *Shearing. The bottom face DC is fixed; a force F along the top face AB moves it to A′B′, a distance AA′ = BB′ = x. The block of height L leans over by the angle φ: its shape changes, not its volume.*

Check 3

Your fingers rub a whiteboard eraser across the board. The stress in the eraser's felt, where it touches the board, is mainly:

a tensile
b compressive
c shearing
d bulk

$$\text{strain} = \frac{\text{change in dimension}}{\text{original dimension}}$$

It is a ratio of two quantities of the same kind, so it has **no unit and no dimensions**. A strain of 0.001 means a change of one part in a thousand, 0.1%.

There are three kinds, one for each kind of change.

i. Longitudinal (tensile or compressive) strain: the change in length per unit original length.

$$\text{longitudinal strain} = \frac{\text{change in length}}{\text{original length}} = \frac{x}{L}$$

A steel wire 2 m long that stretches by 1 mm has a strain of $0.001/2 = 0.0005$, a twentieth of one percent. Metals almost never take a strain of more than a few tenths of a percent before they yield: that is why the extensions in this unit are millimeters.

ii. Volumetric (volume) strain: the change in volume per unit original volume. If a volume V decreases by ΔV,

WORKED EXAMPLE

Energy from a graph

The graph shows the force–extension graph for a spring. Find the elastic potential energy stored when it is stretched from its natural length to 0.50 m.

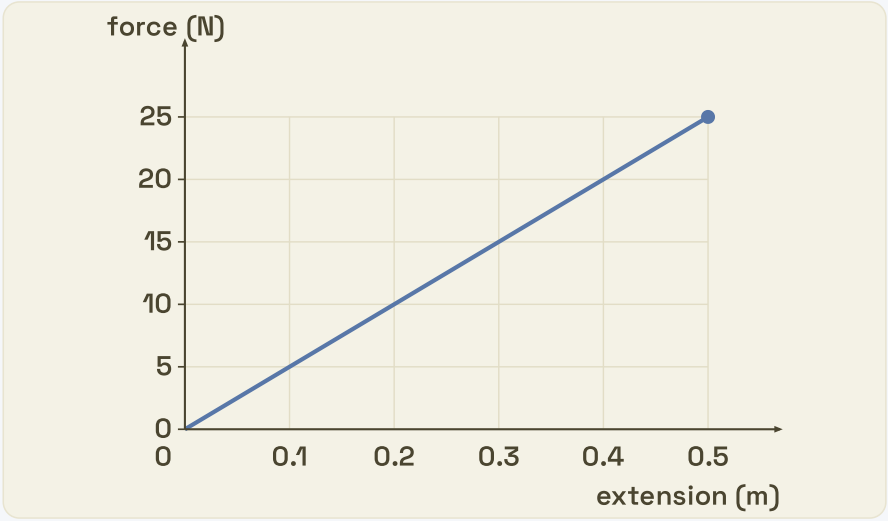


Figure 8.13 *For the example: force against extension for a spring, a straight line from the origin to 25 N at 0.50 m.*

Answer

At an extension of 0.50 m the force is 25 N (read from the graph). The energy stored is the area under the line, a triangle:

$$E = \tfrac{1}{2}Fx = \tfrac{1}{2} \times 25 \times 0.50 = 6.25\ \text{J} \approx 6.3\ \text{J}$$

(The spring's force constant is the slope, $25 \div 0.50 = 50$ N/m.)

YOUR TURN

A catapult's stone

The rubber cord of a catapult has a cross-sectional area of 1 mm² and an unstretched length of 10 cm. It is stretched to 12 cm and released to throw a 5 g stone. Find the speed of the stone (Y for rubber = 5×10^8 N/m²).

Hint

Find the energy stored, $\tfrac{1}{2}\dfrac{YA}{L}x^2$, and give it all to the stone as kinetic energy.

Answer

$x = 0.02$ m, $L = 0.10$ m, $A = 1 \times 10^{-6}$ m².

$$E = \frac{1}{2} \cdot \frac{5 \times 10^8 \times 10^{-6}}{0.10} \times (0.02)^2 = \frac{1}{2} \times 5000 \times 4 \times 10^{-4} = 1.0\ \text{J}$$

$$\tfrac{1}{2}mv^2 = 1.0 \quad \Rightarrow \quad v = \sqrt{\frac{2 \times 1.0}{0.005}} = \sqrt{400} = \mathbf{20\ m/s}$$

WORKED EXAMPLE

The force to stretch a steel wire

What force must be applied to a steel wire 6 m long and 1.6 mm in diameter to stretch it by 1 mm? ($Y = 2.0 \times 10^{11}$ N/m²)

Answer

$L = 6$ m; $d = 1.6 \times 10^{-3}$ m, so $A = \dfrac{\pi d^2}{4} = \dfrac{\pi \times (1.6 \times 10^{-3})^2}{4} = 2.01 \times 10^{-6}$ m²; $x = 1 \times 10^{-3}$ m.

$$F = \frac{YAx}{L} = \frac{2.0 \times 10^{11} \times 2.01 \times 10^{-6} \times 10^{-3}}{6} = 67 \text{ N}$$

About the weight of a 7 kg mass.

WORKED EXAMPLE

One-tenth of one percent

What stress would make a wire 0.1% longer (one-tenth of 1%), if its Young's modulus is 12×10^{10} N/m²? What force gives that stress, if the wire is 0.56 mm in diameter?

Answer

Strain: $\dfrac{x}{L} = \dfrac{1}{10} \times \dfrac{1}{100} = \dfrac{1}{1000} = 1.0 \times 10^{-3}$.

$$\text{stress} = Y \times \text{strain} = 12 \times 10^{10} \times 10^{-3} = 1.2 \times 10^{8} \text{ N/m}^2$$

$$F = \text{stress} \times A = 1.2 \times 10^{8} \times \frac{\pi \times (0.56 \times 10^{-3})^2}{4} = 1.2 \times 10^{8} \times 2.46 \times 10^{-7} = 29.6 \text{ N}$$

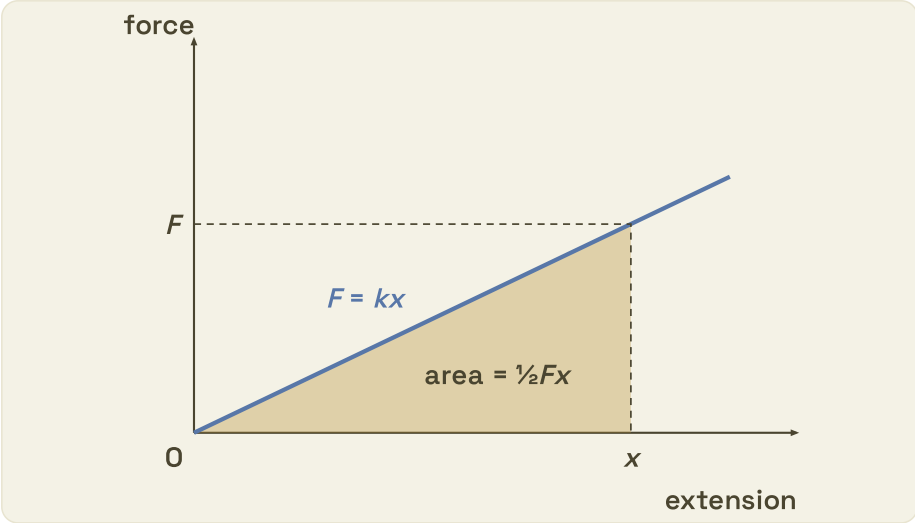


Figure 8.12 *Force against extension within the elastic limit: a straight line, F = kx. The work done in stretching the spring from 0 to x is the shaded area under it, a triangle: ½ × x × F.*

$$W = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2} \times x \times F = \tfrac{1}{2}(kx)x = \tfrac{1}{2}kx^2$$

This is the elastic potential energy stored, the same result as before (with $k = \dfrac{YA}{L}$).

Energy density

The energy stored **per unit volume** of the stretched wire is its **energy density**. The wire's volume is AL, so

$$\text{energy density} = \frac{\frac{1}{2}Fx}{AL} = \frac{1}{2} \cdot \frac{F}{A} \cdot \frac{x}{L} = \frac{1}{2} \times \text{stress} \times \text{strain}$$

and since stress = Y × strain,

$$\text{energy density} = \tfrac{1}{2}Y\,(\text{strain})^2 = \frac{(\text{stress})^2}{2Y}$$

So the energy stored per unit volume is proportional to the **square of the strain**: stretch it twice as far, store four times the energy.

- **A compressed spring dropped into acid.** A spring held squeezed by a thread stores elastic potential energy. If it is dissolved in acid, the energy doesn't vanish: it ends up as extra random motion of the acid's molecules, so the acid gets very slightly warmer than it would with an unsqueezed spring.
- **A catapult.** A stretched catapult stores elastic potential energy. When it is released, the energy goes into the kinetic energy of the stone (exercise question 21).

Check 6

A wire stores 2 J when it is stretched by 1 mm. Within the elastic limit, how much does it store when it is stretched by 3 mm?

a 6 J
b 18 J
c 2 J
d 9 J

$$Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F/A}{x/L} = \frac{F}{x}\cdot\frac{L}{A} \quad \Rightarrow \quad Y = \text{slope} \times \frac{L}{A} \tag{8.6}$$

Why a long, thin wire? Because $x = \dfrac{FL}{AY}$: a long wire stretches more, and a thin one more still, so the tiny extension becomes large enough to measure. The diameter needs the most care: it is small, and it is **squared** in A, so a 1% error in d gives a 2% error in Y.

8.5 Elastic potential energy

When you stretch a wire, you do work against its restoring force. If the wire stays within its elastic limit, this work isn't lost: it is stored in the stretched bonds as **elastic potential energy**, and you get it back when the wire springs back (by the work–energy theorem, Unit 5). A drawn bow, a wound clock spring and a stretched catapult all store energy this way.

The energy stored in a stretched wire

Take a wire of length L and cross-sectional area A, fixed at its top (Figure 8.9). A force at its lower end stretches it slowly by x.

1. **The force grows with the extension.** At an extension x′ (on the way to x), Y = (F/A)/(x′/L), so the force is
$$F = \frac{YA}{L}\,x'$$
2. **Work for a small extra stretch.** Stretching it a little more, by dx′, the force hardly changes, so the work done is
$$dW = F\,dx' = \frac{YA}{L}\,x'\,dx'$$
3. **Add up the work from 0 to x** (integrate):
$$W = \int_0^x \frac{YA}{L}\,x'\,dx' = \frac{YA}{L}\left[\frac{x'^2}{2}\right]_0^x = \frac{1}{2}\,\frac{YA}{L}\,x^2$$
4. **Write it with the final force.** At the end, the force is $F = \dfrac{YA}{L}x$, so $\dfrac{YA}{L}x^2 = Fx$:
$$\textbf{elastic PE} = W = \tfrac{1}{2}Fx = \tfrac{1}{2}\frac{YA}{L}x^2$$

The half is there because the force grows steadily from 0 to F as the wire stretches: on average it is only $\frac{1}{2}F$.

Graphical method

Plot the force F needed to stretch a spring (or wire) against its extension x. Within the elastic limit it is a straight line through the origin whose **slope is the force constant** k (F = kx; the spring's own restoring force is −kx). Equal loads give equal steps of extension.

As in Unit 5, the work done by a force is the **area under its force–extension graph**. Here the area is a triangle:

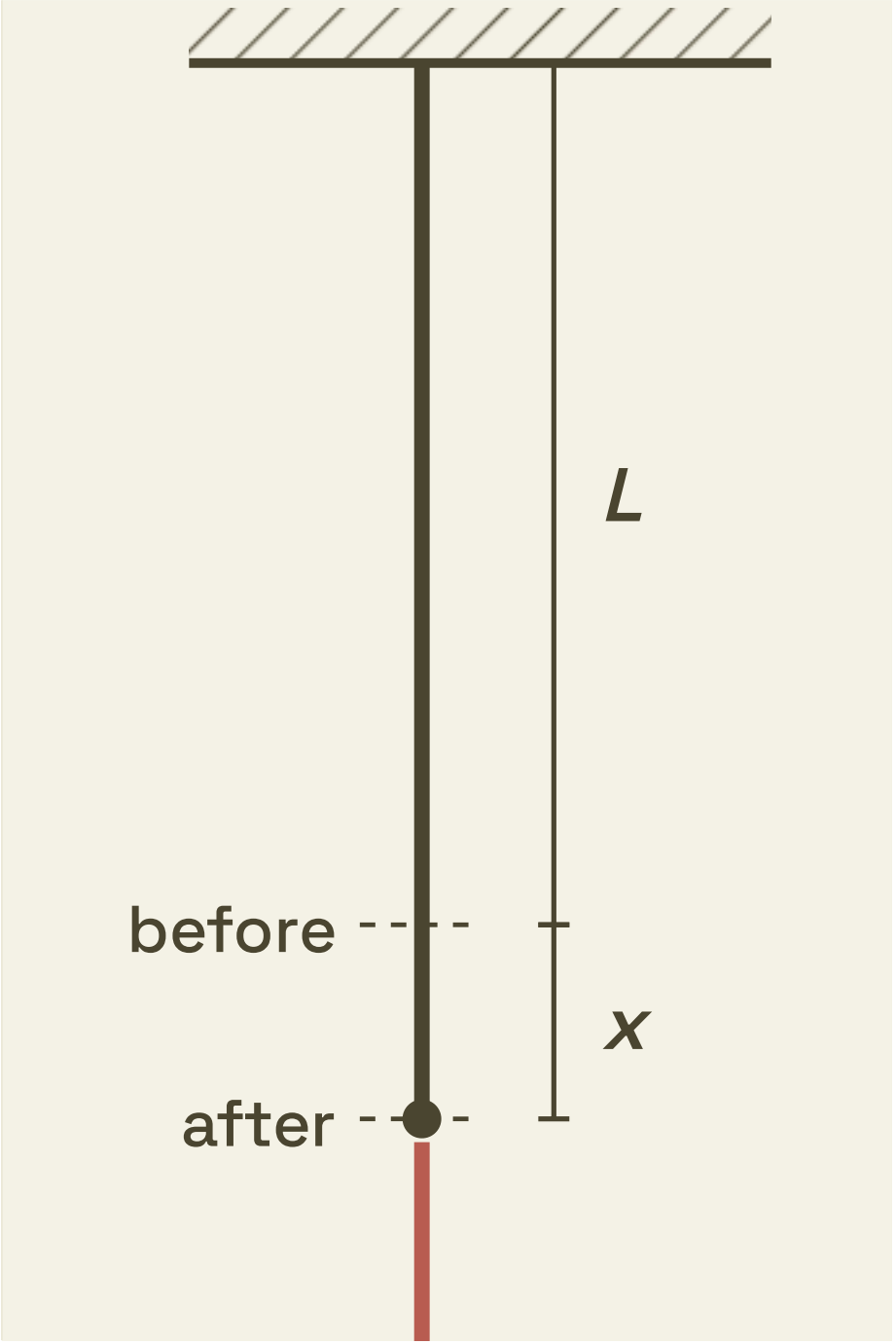




Figure 8.9 *A wire of length L hung from a rigid support. A force F at its lower end stretches it by x (drawn much larger than real). Its longitudinal strain is x/L.*

$$\text{volumetric strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{-\Delta V}{V}$$

The minus sign is there because a pressure from all sides makes the volume **smaller** (ΔV is negative), and we want the strain to be a positive number.

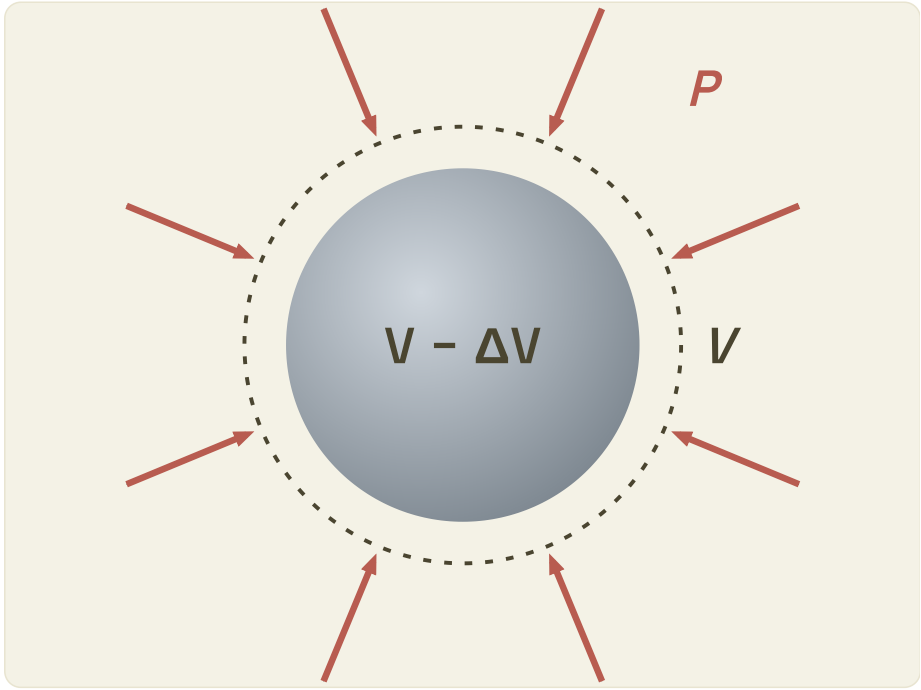


Figure 8.10 *A ball squeezed from all sides by a pressure P: its volume shrinks from V (dotted) to V − ΔV. Its shape doesn't change, only its size.*

iii. Shear strain: under a shearing stress the upper face moves sideways by x relative to the fixed lower face, a distance L below it, and the body leans over by the **angle of shear** φ (Figure 8.8). The shear strain is

$$\text{shear strain} = \tan\phi = \frac{x}{L} \approx \phi \quad (\phi \text{ in radians})$$

For metals, stone and wood φ is tiny (a fraction of a degree), so $\tan\phi \approx \phi$, and the shear strain is simply the angle of shear, in radians.

DID YOU KNOW?

The deepest water is squeezed by only 5%

At the bottom of the Challenger Deep, nearly 11 km down in the Pacific, the water pressure is about 1.1×10^8 Pa, more than a thousand atmospheres. With water's bulk modulus of 2.2×10^9 Pa, its volume shrinks by only about $\frac{1.1 \times 10^8}{2.2 \times 10^9} = 0.05$, five percent. A football taken down there would be crushed to a small fraction of its size; the water around it hardly changes.

c. Modulus of rigidity (shear modulus)

The **modulus of rigidity** (η, eta), or **shear modulus**, is the ratio of the **tangential stress** to the **shear strain** it produces, within the elastic limit. It describes how a material resists a change in **shape** (Figure 8.8).

$$\eta = \frac{\text{tangential stress}}{\text{shear strain}} = \frac{F/A}{\phi} \tag{8.5}$$

where φ = x/L is the angle of shear in radians. Only solids have a modulus of rigidity: a liquid at rest can't hold a shearing stress at all (push sideways on the top of some water and it simply flows). For steel η is about 8×10^{10} N/m², a little less than half its Young's modulus.

WORKED EXAMPLE

Punching a hole

How much force is needed to punch a hole 1 cm in diameter in a steel sheet 5 mm thick, if the steel shears through at a stress (its shearing strength) of 2.76×10^8 N/m²?

Answer

The punch pushes out a round plug of steel. What must be sheared is the **curved side** of the plug, a cylinder of diameter d = 0.01 m and height $t = 5 \times 10^{-3}$ m, not the round face:

$$A = \pi dt = \pi \times 0.01 \times 5 \times 10^{-3} = 1.57 \times 10^{-4}\ \text{m}^2$$

$$F = \text{shearing strength} \times A = 2.76 \times 10^8 \times 1.57 \times 10^{-4} = 4.34 \times 10^4\ \text{N}$$

About **43 kN**, the weight of more than four tonnes, which is why a punch press is driven by a heavy flywheel or a hydraulic ram.

Check 5

Which modulus does a liquid **not** have?

a bulk modulus
b modulus of rigidity
c neither: it has both
d it has no modulus at all

Experimental determination of Young's modulus

The apparatus is the one used to verify Hooke's law (Figure 8.5): two wires A and B, both hung from a rigid support, with small starting loads to take the kinks out. A main scale M is fixed to the reference wire A, and a vernier scale V to the experimental wire B, just beside M.

1. Measure the length L of wire B from the support to the vernier with a meter rule.
2. Measure its diameter d with a screw gauge at several places along it, in two directions at right angles each time, and take the mean. Then $A = \frac{\pi d^2}{4}$.
3. Add loads of 1 kg (or 0.5 kg) one at a time, within the elastic limit, noting the vernier reading after each; then take them off one at a time and note the readings again.
4. Plot the load F against the extension x. The graph is a straight line through the origin; find its slope, $\frac{F}{x}$.

Then, since the tensile stress is $\frac{F}{A}$ and the tensile strain $\frac{x}{L}$,

WORKED EXAMPLE

Stress under a heel

A woman puts her 500 N weight equally on the two heels of her high-heeled shoes. Each heel has an area of 0.750 cm². Find the pressure (compressive stress) each heel puts on the floor.

Answer

Force on each heel: $500 \div 2 = 250$ N. Area: $0.750\ \text{cm}^2 = 0.750 \times 10^{-4}\ \text{m}^2$.

$$P = \frac{F}{A} = \frac{250}{0.750 \times 10^{-4}} = 3.33 \times 10^{6}\ \text{N/m}^2$$

About 33 times atmospheric pressure, and a hundred times or more the stress under her feet in flat shoes. That is why thin heels dent soft wooden floors.

b. Bulk modulus

The **bulk modulus** (B) is the ratio of the **normal stress** (the pressure from all sides) to the **volumetric strain** it produces, within the elastic limit. It describes how a material resists a change in **volume** (Figure 8.10).

If a pressure increase P makes a volume V decrease by ΔV,

$$B = \frac{\text{normal stress}}{\text{volumetric strain}} = \frac{F/A}{-\Delta V/V} = \frac{P}{-\Delta V/V} = -\frac{PV}{\Delta V} \tag{8.4}$$

The minus sign makes B positive, since ΔV is negative when P is positive. Solids, liquids and gases all have a bulk modulus, since all of them resist being squashed (a fluid has no fixed length or shape, so only the bulk modulus applies to it).

The reciprocal of the bulk modulus is the **compressibility**, $K = \frac{1}{B}$: the fractional change in volume per unit increase of pressure, in Pa^{-1}. A material with a large bulk modulus is hard to compress.

Material	Bulk modulus (N/m²)
Steel	about 1.6×10^{11}
Glass	about 4×10^{10}
Water	2.2×10^{9}
Air, at constant temperature	about 1×10^{5} (equal to its pressure)

- Water's bulk modulus is about 20,000 times air's: water is far harder to compress, so in the physicist's sense water is far more elastic than air. That is why hydraulic brakes use oil or brake fluid: an air bubble in the pipes makes the brake pedal spongy.
- Well-compacted concrete has very few air spaces left in it, so it is hard to compress: its compressibility is low and its bulk modulus high.

WORKED EXAMPLE

Squeezing oil

A sample of oil of volume 600 cm³ is put under an extra pressure of 3.6×10^{6} Pa, and its volume decreases by 0.45 cm³. Find the bulk modulus of the oil and its compressibility.

Answer

$V = 600\ \text{cm}^3$, $-\Delta V = 0.45\ \text{cm}^3$: the units cancel in the ratio, so volumetric strain $= \frac{0.45}{600} = 7.5 \times 10^{-4}$.

$$B = \frac{P}{-\Delta V/V} = \frac{3.6 \times 10^{6}}{7.5 \times 10^{-4}} = 4.8 \times 10^{9}\ \text{Pa}$$

$$K = \frac{1}{B} = \frac{1}{4.8 \times 10^{9}} = 2.1 \times 10^{-10}\ \text{Pa}^{-1}$$

Check 4

A rubber block 5.0 cm thick is fixed at its base. A sideways force on its top face moves the top by 2.0 mm. Its shear strain is:

a 2.5 b 0.040
c 0.40 d 25

Elasticity and plasticity

As in Section 8.1, **elasticity** is the property of a body by which it regains its original shape and size after the deforming force is removed.

- A **perfectly elastic body** would regain its original shape and size completely and **immediately** after the force is removed, however large the force. No real body is perfectly elastic; **quartz** fiber and **phosphor bronze** come closest.
- A **perfectly plastic body** doesn't regain its original shape at all, however small the force. **Paraffin wax**, **putty** and **wet clay** are close to perfectly plastic: a potter in Bhaktapur's Pottery Square counts on it.

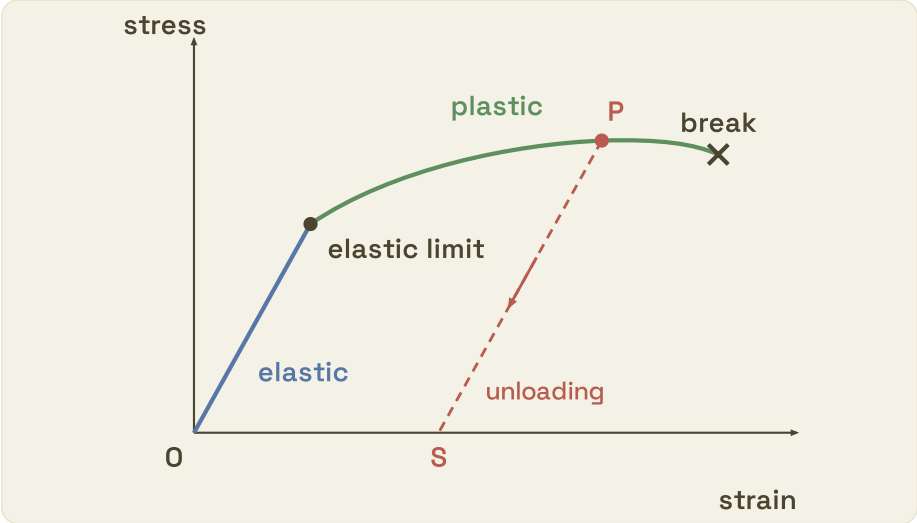


Figure 8.11 *Stress against strain for a metal wire. In the elastic region the graph is a straight line, and taking the load off brings the wire back down the same line to zero. Past the elastic limit the wire is plastic: taken off the load at P, it comes back along a line parallel to the first and is left with a permanent strain OS. Pulled still further, it breaks (×).*

Applications of elasticity

- **Cranes and lifts.** The steel rope of a crane must never be stretched beyond its elastic limit. Knowing the elastic limit of the steel and the greatest load, engineers work out the cross-sectional area the rope needs (the worked example on the lift cable in Section 8.4).
- **How high a mountain can be.** The rock at the base of a mountain carries the weight of all the rock above it. A column of rock of height h and density ρ presses on its base with a stress $\frac{mg}{A} = \frac{\rho A h g}{A} = \rho g h$. When that stress passes the rock's elastic limit, the rock at the bottom yields and flows, and the mountain sinks. So the greatest height is about $h_{\text{max}} = \frac{\sigma_{\text{limit}}}{\rho g}$ (worked out below).
- **Bridges and buildings.** A bridge is designed, from the elasticity of its materials, so that under its own weight, the heaviest traffic and the strongest wind it neither sags too far nor breaks, and every part stays well within its elastic limit.

WORKED EXAMPLE

How high can a mountain be?

Take the elastic limit of typical rock as about 3×10^8 N/m² and its density as 3000 kg/m³. Estimate the greatest height of a mountain on Earth (g = 9.8 m/s²).

Answer

The stress at the base of a column of rock of height h is $\rho g h$. It must stay below the elastic limit:

$$\rho g h_{\max} = \sigma \quad \Rightarrow \quad h_{\max} = \frac{\sigma}{\rho g} = \frac{3 \times 10^8}{3000 \times 9.8} \approx 1.0 \times 10^4 \text{ m}$$

About **10 km**. Sagarmatha, 8,849 m, is close to the limit; a mountain cannot be twice as high. This is only a rough estimate (the rock is not a single upright column, and its strength varies), but it gets the size right. On Mars, where g is only 3.7 m/s², the same rock could stand about 2.6 times as high, and Mars does have the tallest mountain in the solar system, Olympus Mons, about 22 km above Mars's average surface level.

Elastic limit

The **elastic limit** is the greatest stress (or load) a body can take and still regain its original shape and size completely when the stress is removed. Beyond it the body is permanently deformed.

Engineers never load a part up to its elastic limit. They allow only a fraction of it, often one-third or less: the ratio of the elastic limit (or breaking stress) to the greatest stress allowed is the **factor of safety**. It covers flaws in the material, loads larger than expected, wear, rust and fatigue.

Modulus of elasticity

From Hooke's law, stress ∝ strain, so

$$\text{stress} = E \times \text{strain} \qquad \text{or} \qquad E = \frac{\text{stress}}{\text{strain}}$$

E is the **modulus of elasticity** of the material: the stress needed per unit strain. Since strain has no unit, E has the unit of stress, N/m² or Pa. A material with a large modulus needs a large stress for a small strain: it is stiff, or "highly elastic" in the physicist's sense.

8.4 Moduli of elasticity

A body can be stretched (its length changes), squeezed from all sides (its volume changes) or sheared (its shape changes). Each kind of stress and its strain give a modulus.

a. Young's modulus

The **Young's modulus** (Y) of a material is the ratio of the **normal stress** to the **longitudinal strain** it produces, within the elastic limit. It describes how a material resists a change in **length**.

For a wire of length L and cross-sectional area A stretched by x by a force F (Figure 8.9):

$$Y = \frac{\text{normal stress}}{\text{longitudinal strain}} = \frac{F/A}{x/L} \quad \Rightarrow \quad Y = \frac{FL}{Ax} \tag{8.3}$$

Turned round, $x = \dfrac{FL}{AY}$: a wire stretches more if it is longer, thinner, more heavily loaded, or made of a material with a smaller Y. For a round wire of diameter d, $A = \dfrac{\pi d^2}{4}$, so halving the diameter makes the extension four times as large.

Material	Young's modulus (N/m²)
Steel	2.0×10^{11}
Copper	1.1–1.3×10^{11}
Brass	about 1.0×10^{11}
Aluminium	7.0×10^{10}
Glass	about 7×10^{10}
Bone (along its length)	about 1.7×10^{10}
Wood (along the grain)	about 1×10^{10}
Nylon	about 3×10^9
Rubber	about 10^6 to 10^7

DID YOU KNOW?

Thomas Young, who did everything

The modulus is named after **Thomas Young** (1773–1829), an English doctor who in 1801 showed that light is a wave (his double-slit experiment, Grade 12), worked out how the eye focuses and how we see color, and helped read the Egyptian hieroglyphs on the Rosetta Stone. He described the modulus of elasticity in his lectures of 1807. He was also the first to use the word *energy* in its modern sense.

Two things to know about Young's modulus:

- **It falls as the temperature rises.** In a hot metal the atoms vibrate harder and sit a little farther apart, where the bonds pull back less strongly for the same extra stretch. So the same stress gives a larger strain, and Y is smaller.
- **A wire stretches under its own weight.** A uniform wire of mass m, length L, area A and Young's modulus Y hanging from a ceiling stretches by $\dfrac{mgL}{2AY}$, half of what it would stretch if its whole weight hung from its lower end. The derivation is in exercise question 9.

WORKED EXAMPLE

A lift cable

A steel cable of cross-sectional area 3.00 cm² has an elastic limit of 2.40×10^8 Pa. Find the greatest upward acceleration that can be given to a 1200 kg lift (elevator) hanging from it, if the stress is not to exceed one-third of the elastic limit. (g = 9.8 m/s²)

Answer

Area $A = 3.00 \text{ cm}^2 = 3.00 \times 10^{-4} \text{ m}^2$.

Greatest stress allowed: $\dfrac{F}{A} = \dfrac{1}{3} \times 2.40 \times 10^8 = 8.0 \times 10^7$ Pa, so the greatest tension is

$$F = 8.0 \times 10^7 \times 3.00 \times 10^{-4} = 2.4 \times 10^4 \text{ N}$$

For the lift accelerating upward (Newton's second law, Unit 4): $F - mg = ma$, so

$$a = \frac{F - mg}{m} = \frac{24\,000 - 1200 \times 9.8}{1200} = \frac{24\,000 - 11\,760}{1200} = \mathbf{10.2 \text{ m/s}^2}$$

(With g = 10 m/s², 10.0 m/s².) Real lifts accelerate at under 2 m/s², so this cable has a large margin.