Unit 8

Answers: Elasticity

Quick checks

1 b Beyond the elastic limit the deformation is partly permanent: layers of atoms have slipped into new places, so the wire keeps a **permanent set** and stays longer. It doesn't break at once (copper can stretch a long way before it breaks), and it certainly doesn't get shorter.

2 a $k = F/x$: the same F over a smaller x gives a larger k. The stiffer spring is the one that stretches less. Its length doesn't matter here: k already includes it.

3 c The board's friction acts **along** the felt's face while your hand pushes the top the other way: forces parallel to the surface are shearing. (You also press down a little, which adds some compressive stress, but the rubbing is shear.)

4 b Shear strain = x/L = 2.0 mm ÷ 50 mm = 0.040 (about 2.3°). 0.40 forgets to change cm to mm; 2.5 and 25 divide the wrong way round. A strain has no unit, so the two lengths must be in the same unit.

5 b A liquid resists being squeezed into a smaller volume, so it has a bulk modulus. But it can't keep a shearing stress: it flows. So it has no modulus of rigidity (and no Young's modulus, since it has no length of its own to stretch).

6 b Energy $= \frac{1}{2}kx^2$, proportional to x^2: three times the extension stores $3^2 = 9$ times the energy, $9 \times 2 = 18$ J. 6 J forgets that the force also triples.

7 c $\dfrac{\Delta V}{V} = \varepsilon(1 - 2\sigma) = 0.002 \times (1 - 0.5) = 0.001$, an **increase** of 0.1%. It would stay the same only for $\sigma = 0.5$; 0.2% is the change in length alone, forgetting that the wire gets thinner.

8 c The curve is highest at D, the ultimate tensile stress. After it the wire necks and breaks at a **lower** load, at E. The elastic limit and the yield point come much earlier, at far lower stress.

9 b All three resist being squeezed into a smaller volume, so all have a **bulk modulus**. Only solids have a shear modulus (fluids flow under a shearing stress) and a Young's modulus (fluids have no length of their own to stretch).

10 b The mass falls until it stops for an instant at the lowest point, where all the potential energy it has lost is stored in the wire: $mgx = \frac{1}{2}kx^2$, so $x = \dfrac{2mg}{k}$. That is twice the stretch $\dfrac{mg}{k}$ at which it would hang at rest (if lowered gently); released suddenly, it overshoots and oscillates about that point.

11 b Poisson's ratio is the lateral strain $\frac{\Delta R}{R}$ divided by the longitudinal strain $\frac{\Delta L}{L}$: $\frac{\Delta R}{R} \div \frac{\Delta L}{L} = \frac{\Delta R}{R} \times \frac{L}{\Delta L}$. (The diameter's fractional change is the same as the radius's.)

12 a Heated, the rod tries to expand, but the supports stop it. They push inwards on its ends, squeezing it back to its original length: the stress is **compressive**. (If it were fixed and then cooled, it would try to shrink, and the stress would be tensile.)

13 d In physics, the most elastic material is the one with the largest modulus: the one that needs the greatest stress for a given strain. Steel's Young's modulus (2×10^{11} N/m²) is the largest here; rubber and sponge deform easily and are the least elastic.

14 b At a higher temperature the atoms vibrate more and sit a little farther apart, where the forces between them pull back less strongly for each extra stretch. The material becomes less stiff: Y decreases.

15 d Elastomers, such as rubber, take huge strains (several hundred percent) and return. Ductile metals stretch a long way too, but **plastically**: they don't come back. Brittle materials hardly stretch at all.

16 c Shearing makes the layers of the body slide over one another, so a rectangle becomes a parallelogram: the **shape** changes, not the volume. (Length is changed by a tensile or compressive stress, volume by a pressure from all sides.)

17 b Take a short piece dy of the wire at a height y above its lower end. It holds the weight of the wire below it, $\frac{y}{L}mg$, so it stretches by $d() = \frac{(mgy/L)\,dy}{AY}$. Adding up along the wire:

$$= \int_0^L \frac{mg\,y}{LAY}\,dy = \frac{mg}{LAY} \cdot \frac{L^2}{2} = \frac{mgL}{2AY}$$

Half of $\frac{mgL}{AY}$, the stretch if its whole weight hung from its end: on average each part holds half the weight.

18 a Shear strain = x/L = tan φ ≈ φ, the angle of shear in radians. (The angle of twist is how far the end of a twisted rod turns, which is a different angle; the volume doesn't change in shear.)

19 d $P = 100 \times 1.013 \times 10^5 = 1.013 \times 10^7$ Pa; volumetric strain $= 0.01\% = 1 \times 10^{-4}$.

$$B = \frac{P}{-\Delta V/V} = \frac{1.013 \times 10^7}{1 \times 10^{-4}} \approx 1 \times 10^{11}\ \text{N/m}^2.$$

20 d That delay is the **elastic after-effect**. Elastic fatigue is the loss of strength after many cycles of loading.

13.

a. **Hooke's law:** within the elastic limit, the extension of a body is directly proportional to the load (force) applied to it; equivalently, stress is directly proportional to strain.

b. **No.** It holds only up to the proportional limit, and not for every material even there. Metals obey it for small strains; brittle materials such as glass obey it until they break; but **elastomers** such as rubber don't obey it at all: their stress–strain graph curves from the start, though they spring back.

c. At the elastic limit, strain = stress ÷ Y:

$$\frac{x}{L} = \frac{8 \times 10^{8}}{2 \times 10^{11}} = 4 \times 10^{-3} \quad \Rightarrow \quad x = 4 \times 10^{-3} \times 0.5 = 2 \times 10^{-3}$$

The greatest elongation is **2 mm**.

14.

a. **Young's modulus** (for a change in length), **bulk modulus** (for a change in volume) and **modulus of rigidity** or shear modulus (for a change in shape).

b. Only **volume (bulk) elasticity**. Water resists being compressed, so it has a bulk modulus (2.2×10^{9} N/m²), but it can't hold a shearing stress or a length of its own, so it has no modulus of rigidity and no Young's modulus.

c. $A = \dfrac{\pi d^2}{4} = \dfrac{\pi \times (1.0 \times 10^{-3})^2}{4} = 7.85 \times 10^{-7}$ m².

$$x = \frac{FL}{AY} = \frac{2000 \times 0.80}{7.85 \times 10^{-7} \times 2.0 \times 10^{11}} = \frac{1600}{1.57 \times 10^{5}} = 1.02 \times 10^{-2}$$

About **1.0 cm**. (The stress, 2.5×10^{9} N/m², is about as much as the strongest piano wire can take: a real guitar string is tightened to only about 100 N.)

15.

a. When a wire is stretched (or a rod compressed), its width changes as well as its length. Within the elastic limit, the ratio of the **lateral strain** (the fractional change in diameter) to the **longitudinal strain** (the fractional change in length) is a constant for the material, **Poisson's ratio** σ: $\sigma = \dfrac{\Delta D / D}{\Delta L / L}$. It has no unit.

b. **Elastic after-effect** is the delay with which a body regains its original shape and size after the deforming force has been removed.

c. $F = mg = 5 \times 10 = 50$ N.

Compression: $x = \dfrac{FL}{AY} = \dfrac{50 \times 0.5}{1 \times 10^{-4} \times 1.0 \times 10^{11}} = \dfrac{25}{10^{7}} = 2.5 \times 10^{-6}$ m.
Energy stored: $E = \frac{1}{2} F x = \frac{1}{2} \times 50 \times 2.5 \times 10^{-6} = 6.25 \times 10^{-5}$ J.

16. a. **Elastic potential energy** is the energy stored in a body by deforming it within its elastic limit: the work done against its restoring force, given back when it returns to its shape.

Derivation (Section 8.5): a wire of length L and area A, fixed at its top, is stretched slowly. At extension x′ the force is $F = \dfrac{YA}{L}x'$ (from Y = (F/A)/(x′/L)). For a further small extension dx′ the work is $dW = \dfrac{YA}{L}x'\,dx'$. Integrating from 0 to the final extension x:

$$W = \int_0^x \frac{YA}{L}x'\,dx' = \frac{1}{2}\frac{YA}{L}x^2$$

Since the final force is $F = \dfrac{YA}{L}x$, $W = \tfrac{1}{2}Fx$. So the **elastic PE = ½Fx = ½(YA/L)x²**, and per unit volume, ½ stress × strain.

b. $\dfrac{\Delta V}{V} = \varepsilon(1-2\sigma) = 4\times10^{-3}\times(1-0.4) = 2.4\times10^{-3}$, an increase of **0.24%** (if the rod is stretched; a decrease if it is compressed).

c. $A = 0.01^2 = 1\times10^{-6}$ m².

Longitudinal strain: $\dfrac{F}{AY} = \dfrac{25}{1\times10^{-6}\times1.1\times10^{11}} = 2.27\times10^{-4}$.

Lateral strain: $\dfrac{\Delta r}{r} = \sigma\times2.27\times10^{-4} = 0.32\times2.27\times10^{-4} = 7.27\times10^{-5}$.

Since $A = \pi r^2$, $\dfrac{\Delta A}{A} = 2\dfrac{\Delta r}{r} = 1.45\times10^{-4}$, so

$$\Delta A = 1.45\times10^{-4}\times1\times10^{-6} = 1.45\times10^{-102}$$

17. a. Within the elastic limit, each kind of stress and the strain it causes have a constant ratio, a modulus of elasticity:

- **Young's modulus**, $Y = \quad = \dfrac{FL}{Ax}$: resistance to a change in length; for solids only.
- **Bulk modulus**, $B = \quad = \dfrac{P}{-\Delta V/V}$: resistance to a change in volume; solids, liquids and gases all have one. Its reciprocal is the compressibility.
- **Modulus of rigidity**, $\eta = \quad = \dfrac{F/A}{\phi}$: resistance to a change in shape; for solids only.

All three are in N/m² (Pa).

b. Doubling the length means an extension x = L, a strain of 1:

$$F = \frac{YAx}{L} = YA \times 1 = 2.0 \times 10^{11} \times 1 \times 10^{-4} = 2.0 \times 10^{7}$$

So the Young's modulus is the stress that would double a wire's length, if Hooke's law held that far. It doesn't: real steel yields at a strain of a fraction of one percent and breaks long before its length doubles.

c. The tension F is the same in both (they are in a line), and so are the areas. Equal extensions:

$$\frac{FL_s}{AY_s} = \frac{FL_c}{AY_c} \quad \Rightarrow \quad \frac{L_s}{L_c} = \frac{Y_s}{Y_c} = \frac{2.0 \times 10^{11}}{1.1 \times 10^{11}} = \frac{20}{11}$$

The steel wire must be the longer, in the ratio **20 : 11**.

18. a. The **elastic limit** is the greatest stress a body can take and still return completely to its original shape and size when the stress is removed. The **breaking stress** is the stress at which it breaks; the greatest stress it can bear (the ultimate tensile stress, point D of the stress–strain curve) is often given as the breaking stress of the material.

b. **Elastic hysteresis** is the lag of strain behind stress when a body is loaded and then unloaded: the stress–strain curve on unloading lies below the curve on loading, making a closed loop whose area is the energy turned into heat per unit volume in each cycle.

c. i. $A = \pi r^2 = \pi \times (0.25 \times 10^{-3})^2 = 1.96 \times 10^{-7} \approx 2.0 \times 10^{-7}$ m².

ii. $W = \quad \times A = 2.0 \times 10^{8} \times 1.96 \times 10^{-7} = 39$ N.

iii. $\text{Strain} = \dfrac{x}{L} = \dfrac{0.050}{1.0} = 0.05.$

iv. $Y = \quad = \dfrac{2.0 \times 10^{8}}{0.05} = 4.0 \times 10^{9}$ Pa. (Nylon's modulus is about a fiftieth of steel's, which is why a fishing line stretches so much.)

19. $L = 1.00$ m, $A = 0.030^2 = 3.0 \times 10^{-6}$ m², F = 100 N.

$$x = \frac{FL}{AY} = \frac{100 \times 1.00}{3.0 \times 10^{-6} \times 2.0 \times 10^{11}} = 1.67 \times 10^{-4}$$

$$W = \tfrac{1}{2}Fx = \tfrac{1}{2} \times 100 \times 1.67 \times 10^{-4} = 8.3 \times 10^{-3}$$

20. Volume: $V = \dfrac{m}{\rho} = \dfrac{0.016}{7800} = 2.05 \times 10^{-6}$ m³. Area: $A = \dfrac{V}{L} = \dfrac{2.05 \times 10^{-6}}{2.50} = 8.21 \times 10^{-7}$ m².

a. $Y = \dfrac{FL}{Ax} = \dfrac{80 \times 2.50}{8.21 \times 10^{-7} \times 1.2 \times 10^{-3}} = 2.0 \times 10^{11}$ N/m².

b. $E = \tfrac{1}{2}Fx = \tfrac{1}{2} \times 80 \times 1.2 \times 10^{-3} = 4.8 \times 10^{-2}$ J.

21. $x = 2.0$ cm = 0.020 m, $L = 0.100$ m, $A = 1.0 \times 10^{-6}$ m².

Energy stored:

$$E = \frac{1}{2}\frac{YA}{L}x^2 = \frac{1}{2} \times \frac{5.0 \times 10^{8} \times 1.0 \times 10^{-6}}{0.100} \times (0.020)^2 = 1.0$$

All of it becomes the stone's kinetic energy:

$$\tfrac{1}{2} \times 0.0050 \times v^2 = 1.0 \quad \Rightarrow \quad v = \sqrt{400} =$$

(In practice a little is lost to the cord's hysteresis and to its own motion, so the stone goes a little slower.)

22. With the same F and A, $x \propto \dfrac{L}{Y}$:

$$\frac{x_c}{x_s} = \frac{L_c}{Y_c} \cdot \frac{Y_s}{L_s} = \frac{1}{1.2} \times \frac{2.0}{2} = \frac{1}{1.2} = \frac{5}{6}$$

So $x_c = \tfrac{5}{6}x_s$, and $x_c + x_s = 1$ cm:

$$\tfrac{11}{6}x_s = 1 \quad \Rightarrow \quad x_s = 0.545, \quad x_c = 0.455$$

The copper wire stretches by about **0.0045 m** and the steel wire by about **0.0055 m**.

23. $V = \dfrac{0.020}{8000} = 2.5 \times 10^{-6}$ m³, so $A = \dfrac{V}{L} = \dfrac{2.5 \times 10^{-6}}{2.5} = 1.0 \times 10^{-6}$ m².

$$Y = \frac{FL}{Ax} = \frac{80 \times 2.5}{1.0 \times 10^{-6} \times 1.0 \times 10^{-3}} = 2.0 \times 10^{112}$$

$$E = \tfrac{1}{2}Fx = \tfrac{1}{2} \times 80 \times 1.0 \times 10^{-3} = 4.0 \times 10^{-2}$$

24. $F = 15 \times 9.8 = 147$ N.

Stress: $\dfrac{F}{A} = \dfrac{147}{1.0 \times 10^{-6}} = 1.47 \times 10^{8}$ N/m².

Extension: $x = \dfrac{\times L}{Y} = \dfrac{1.47 \times 10^{8} \times 3.0}{2.0 \times 10^{11}} = 2.2 \times 10^{-3}$ m.

Energy: $E = \tfrac{1}{2}Fx = \tfrac{1}{2} \times 147 \times 2.205 \times 10^{-3} = 0.162$ J.

25. From $x = \dfrac{FL}{AY}$, the least area is

$$A = \frac{FL}{Yx} = \frac{400 \times 2.00}{2.0 \times 10^{11} \times 0.0025} = 1.6 \times 10^{-62}$$

$$d = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{4 \times 1.6 \times 10^{-6}}{\pi}} = 1.43 \times 10^{-3}$$

At least **1.43 mm**.

26. $F = 5 \times 9.8 = 49$ N; $A = \pi \times (0.01)^2 = 3.14 \times 10^{-4}$ m²; $L = 0.50$ m.

a. $x = \dfrac{FL}{AY} = \dfrac{49 \times 0.50}{3.14 \times 10^{-4} \times 3.5 \times 10^{10}} = \dfrac{24.5}{1.10 \times 10^{7}} = 2.2 \times 10^{-6}$ m.

b. $E = \frac{1}{2}Fx = \frac{1}{2} \times 49 \times 2.23 \times 10^{-6} = 5.5 \times 10^{-5}$ J.

(With g = 10 m/s², 2.27×10^{-6} m and 5.7×10^{-5} J.)

27. $A = \dfrac{\pi d^2}{4} = \dfrac{\pi \times (1.84 \times 10^{-3})^2}{4} = 2.66 \times 10^{-6}$ m².

$$= \frac{F}{A} = \frac{90.8}{2.66 \times 10^{-6}} = 3.41 \times 10^{7}$$

28. $A = 0.50^2 = 5.0 \times 10^{-5}$ m²; $x = 2.0 \times 10^{-3}$ m.

$$Y = \frac{FL}{Ax} = \frac{5000 \times 4.00}{5.0 \times 10^{-5} \times 2.0 \times 10^{-3}} = \frac{20\,000}{1.0 \times 10^{-7}} = 2.0 \times 10^{112}$$

29. Volumetric strain: $\dfrac{0.45}{600} = 7.5 \times 10^{-4}$.

$$B = \frac{P}{-\Delta V/V} = \frac{3.6 \times 10^{6}}{7.5 \times 10^{-4}} = 4.8 \times 10^{9}$$

$$K = \frac{1}{B} = 2.1 \times 10^{-10-1}$$