Unit 7

Gravitation

This unit online, with its interactives:
science-playground.com/s/11f7

A mango drops from its branch in Bhaktapur. Some 384,000 km away, the Moon swings round the Earth on its monthly circle. A television satellite hangs, apparently motionless, in the southern sky above Nepal, and the dish on your roof never has to move to follow it. Isaac Newton's great idea was that all three are the **same force**: the pull every piece of matter has on every other.

In this unit you will write that pull as a law, use it to weigh the Earth, follow how gravity changes as you climb Everest or dig a mine, measure the energy it stores, and work out how fast a satellite must go to stay up, and how fast a rocket must go to get away for good. At the end you will see how a phone uses satellites and the speed of light to find where you are standing.

7.1 Introduction

People argued about why things fall for two thousand years. Aristotle (384–322 BCE) thought heavy things fall because they belong at the center of the universe; medieval scholars in the Islamic world and in Europe added ideas of their own; and in the 1600s Galileo, Kepler and Robert Hooke came close to the answer. Isaac Newton (1642–1727) put it together: the story says he wondered, when an apple fell, whether the pull that brought it down reached all the way to the Moon. The word **gravity** comes from the Latin *gravitas*, "weight". Newton did not discover that things fall; he discovered **why**, and that the same law rules the sky.

The force of attraction between **any two bodies** because of their masses is called **gravitation**. When one of the two is the Earth, the force is called **gravity**. So gravity is one case of gravitation: the pull of the Earth on a mango is gravity, the pull between the Sun and Jupiter is gravitation.

Gravitation is the **weakest** of the four fundamental forces of nature: the electric force between a proton and an electron is about 10^{39} times stronger than the gravitational force between them. It rules the sky only because it always attracts and never cancels, while electric charges come in two kinds that cancel out in bulk matter.

7.2 Newton's law of gravitation

Newton published his law in 1687, in the *Principia*:

> Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

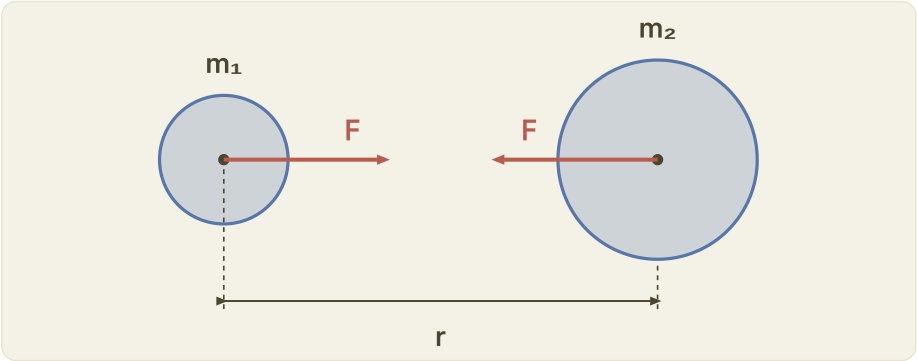


Figure 7.1 *Two masses m_1 and m_2, their centers a distance r apart, pull each other with equal and opposite forces F, along the line joining their centers.*

G is the **universal gravitational constant**:

$$G = 6.67 \times 10^{-11}\ \text{N m}^2/\text{kg}^2$$

What G means: put $m_1 = m_2 = 1$ kg and $r = 1$ m; then $F = G$. So G **is the force of attraction between two masses of 1 kg each, placed 1 m apart**: 6.67×10^{-11} N, about the weight of a speck of dust.

- **Universal:** G has the same value everywhere and for all bodies, a stone, a planet or a star, whatever they are made of and whatever lies between them. Nothing can shield a body from gravity.
- The forces form an **action–reaction pair** (Newton's third law): the Earth pulls a mango down exactly as hard as the mango pulls the Earth up.
- For spheres (the Earth, the Moon, a ball), r is the distance between their **centers**: a uniform sphere pulls things outside it as if all its mass were at its center.

A STORY FROM SCIENCE

Weighing the Earth in a shed

Newton never knew the value of G: the pull between ordinary objects is far too small to feel. In 1798 Henry Cavendish measured it in an outbuilding of his London house. Two small lead balls on the ends of a rod hung from a thin wire; two big lead balls, 158 kg each, were brought beside them. Their pull turned the rod a tiny angle, twisting the wire, and from the twist he found the force. He watched through a telescope from outside the room, so that the warmth of his body would not set up air currents. Knowing G, he could work out the mass of the Earth (7.3), which is why his experiment is called "weighing the Earth".

YOUR TURN

The pull between two people

Two students, 50 kg and 60 kg, sit 1 m apart. How strongly do they attract each other?

Answer

$$F = G\frac{m_1 m_2}{r^2} = \frac{6.67 \times 10^{-11} \times 50 \times 60}{1^2} = 2.0 \times 10^{-7}\ \text{N}$$

About the weight of a grain of sand a quarter of a millimeter across: far too small to notice beside friction.

Check 1

The distance between two bodies is halved and one mass is doubled. The gravitational force becomes:

a 2 times
b 4 times
c 8 times
d the same

7.3 Acceleration due to gravity

The acceleration of a body falling freely under the Earth's gravity alone is the **acceleration due to gravity**, g. Near the Earth's surface it is about **9.8 m/s²**, the same for every body, whatever its mass, size or shape (in a vacuum, a feather and a hammer fall together). A ball thrown straight up loses 9.8 m/s of speed every second; a falling ball gains 9.8 m/s every second.

Derivation: $g = GM/R^2$

Take a body of mass m on the surface of the Earth, mass M and radius R, treated as a uniform sphere.

1. By Newton's law of gravitation, the Earth pulls it with $F = \dfrac{GMm}{R^2}$.
2. By Newton's second law, this force gives it an acceleration g: $F = mg$.
3. So $mg = \dfrac{GMm}{R^2}$, and m cancels:

$$\boxed{g = \frac{GM}{R^2}}$$

The body's mass has canceled: that is **why** all bodies fall with the same acceleration. And g depends only on the planet's mass and radius, so for a planet $g \propto M/R^2$.

The Earth is not a perfect sphere: spinning has made it bulge at the equator, where the surface is about 21 km farther from the center than at the poles, and the spin itself lessens the pull a body feels there (see below). So g varies over the Earth:

- at the poles, $g = 9.83$ m/s²;
- at the equator, $g = 9.78$ m/s².

30. Two stars of masses 10^{20} kg and 2×10^{20} kg, 10^6 m apart, go round their common center of mass with angular velocity ω. The only force on each is their gravitational pull on each other. Find ω. ($G = 6.7 \times 10^{-11}$ N m²/kg²)
31. On its way to the Moon, Apollo 11 was first put in a circular "parking orbit" 189 km above the Earth's surface. Taking the gravitational field strength in this orbit as 9.4 N/kg, calculate i. the spacecraft's speed and ii. the time for one orbit. (Radius of the Earth 6370 km)
32. Explorer 38, a radio astronomy satellite of mass 200 kg, circles the Earth in an orbit of average radius 3R/2, where R is the Earth's radius. Taking the gravitational pull on 1 kg at the Earth's surface as 10 N, calculate the pull on the satellite.
33. The Earth is flattened, with a polar radius of 6.357×10^6 m and an equatorial radius of 6.378×10^6 m. Find the difference Δg between the acceleration of free fall at a pole and at the equator due to this difference in radii alone. What other factors affect Δg? (Mass of the Earth 5.957×10^{24} kg, $G = 6.670 \times 10^{-11}$ N m²/kg²)
34. Find g from the motion of the Moon, given that it goes round the Earth in 27 days 8 hours in an orbit of radius 60.1 times the radius of the Earth. (Radius of the Earth 6.36×10^6 m)
35. The Moon's orbit is approximately a circle of radius 60 times the Earth's equatorial radius. Find the time the Moon takes to complete one orbit, neglecting the rotation of the Earth. (g at the Earth's poles 9.8 m/s², equatorial radius 6.4×10^6 m, 1 day $= 8.6 \times 10^4$ s)
36. Two particles of masses 1 kg and 2 kg are at x = 0 and x = 3 m. Find the position of their center of mass.
37. With what velocity must a satellite be launched at a height of 600 km above the Earth's surface to move in a circular orbit? ($G = 6.67 \times 10^{-11}$ N m²/kg², radius of the Earth 6.4×10^6 m, mass of the Earth 5.98×10^{24} kg)

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

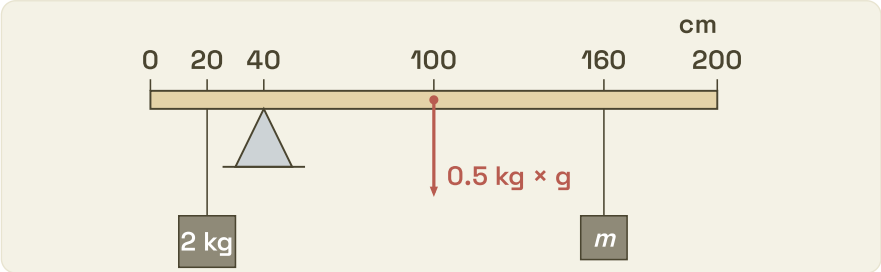


Figure 7.12 *For question 20: a uniform rod 200 cm long on a wedge at the 40 cm mark, with 2 kg at 20 cm and an unknown m at 160 cm. Its own weight acts at its middle.*

21.
   a. How does GPS find a position?
   b. What is the least number of satellites needed to find one's position with GPS?
   c. Name the global navigation satellite systems that can be used instead of GPS.

22.
   a. Define acceleration due to gravity. Derive an expression showing how it varies with altitude.
   b. At what height above the Earth's surface is g reduced by 36% of its value at the surface? (Radius of the Earth 6400 km)

23.
   a. Estimate your weight at the center of the Earth.
   b. Find an expression for how the acceleration due to gravity changes with depth below the Earth's surface.
   c. Calculate the force of the Earth's gravity on a 75 kg astronaut repairing the Hubble Space Telescope, 600 km above the Earth's surface. (Radius of the Earth 6400 km, g = 9.8 m/s²)
   d. Compare this force with the astronaut's weight on the Earth's surface.

24. An object launched from the Earth's surface fast enough rises into space and never returns; the least such speed is the Earth's escape velocity.
   a. Why must this velocity be so large?
   b. Derive an expression for it at the Earth's surface.
   c. The Earth has 9 times the mass and twice the radius of the planet Mars. Find the least velocity a rocket needs to escape the gravity of Mars. (Escape velocity at the Earth's surface 11.2 km/s)

25. A satellite in a parking orbit has a period of 24 hours, equal to that of the Earth's rotation.
   a. Why is a satellite in this orbit called a geostationary satellite?
   b. Derive an expression for the height of the parking orbit above the Earth's surface.
   c. An artificial satellite orbits the Earth in a circle 300 km above its surface. Find its orbital speed and its period. (Radius of the Earth 6400 km, g = 980 cm/s²)

26.
   a. Distinguish between gravitational potential and gravitational potential energy.
   b. Obtain the formula for the gravitational potential energy of a body on the surface of the Earth.
   c. In an otherwise empty universe, two planets, each of mass 2×10^{25} kg and radius 10^6 m, are 10^{12} m apart and nearly at rest. What are their speeds when they collide?

27.
   a. What is the orbital velocity of a satellite?
   b. Find expressions for the orbital velocity and the period of a satellite.
   c. An Earth satellite moves in a circular orbit at 6200 m/s. Find i. the time of one revolution and ii. its radial (centripetal) acceleration. ($G = 6.67 \times 10^{-11}$ N m²/kg², mass of the Earth 5.98×10^{24} kg)

Numerical problems

28. Find the period of a satellite in a circular orbit 3.6×10^6 m above the Earth's surface. Take the Earth as a uniform sphere of radius 6.4×10^6 m and mass 6×10^{24} kg, and $G = 6.7 \times 10^{-11}$ N m²/kg².

29. Taking the mean density of the Earth as 5500 kg/m³, $G = 6.7 \times 10^{-11}$ N m²/kg² and the Earth's radius as 6400 km, find the acceleration of free fall at the Earth's surface.

CAREFUL

Steep slopes and gravity

Water rushes down a steep stream, and running downhill is hard to stop and tiring on the knees. That is gravity, but not because g changes: on a slope at angle θ, the part of your weight along the slope, $mg\sin\theta$, is not balanced, so it speeds you up, and your legs must supply the braking force. The value of g at the top and bottom of a hill differs by a fraction of a percent.

Weighing the Earth

Rearranged, $g = GM/R^2$ gives the mass of the Earth from three things we can measure: g (with a pendulum), R (Eratosthenes did it in about 240 BCE; see the project at the end), and G (Cavendish).

$$M = \frac{gR^2}{G}$$

Dividing by the volume $\frac{4}{3}\pi R^3$ gives the Earth's mean density, about 5,500 kg/m³: twice that of surface rocks, so the inside must be much denser (it has an iron core).

WORKED EXAMPLE

The mass of the Earth

The acceleration of free fall at the Earth's surface is 9.8 m/s² and the radius of the Earth is 6400 km. Find the mass of the Earth. ($G = 6.7 \times 10^{-11}$ N m²/kg²)

Answer

$g = 9.8$ m/s², $R = 6400$ km $= 6.4 \times 10^6$ m, $G = 6.7 \times 10^{-11}$ N m²/kg².

$$M = \frac{gR^2}{G} = \frac{9.8 \times (6.4 \times 10^6)^2}{6.7 \times 10^{-11}} = \frac{9.8 \times 4.096 \times 10^{13}}{6.7 \times 10^{-11}} = 5.99 \times 10^{24}\ \text{kg}$$

(The accepted value is 5.97×10^{24} kg.)

WORKED EXAMPLE

A satellite at twice the Earth's radius

The gravitational force on a 1 kg body at the Earth's surface is 10 N. Treating the Earth as a sphere of radius R, find the gravitational force on a satellite of mass 100 kg in a circular orbit of radius $2R$ from the center of the Earth.

Answer

At the surface, on 1 kg: $\frac{GM \times 1}{R^2} = 10$ N, so $\frac{GM}{R^2} = 10$ N/kg.

On the satellite, $m = 100$ kg at $r = 2R$:

$$F = \frac{GMm}{(2R)^2} = \frac{GM}{R^2} \times \frac{100}{4} = 10 \times 25 = 250\ \text{N}$$

Twice as far from the center, the pull per kilogram is a quarter: 2.5 N/kg instead of 10.

Things worth knowing about gravity

- **The Moon falls, and misses.** The Earth's pull on the Moon is at right angles to the Moon's velocity: it does no work, changes only the Moon's direction, and provides the centripetal force for its nearly circular orbit (Unit 6). The Moon is falling towards the Earth all the time, but it moves sideways fast enough that it keeps missing.
- **A satellite's speed stays the same**, in a circular orbit, for the same reason: the pull is always at right angles to its motion, so it turns the satellite without speeding it up or slowing it down. The field certainly acts on it: without the pull, the satellite would fly off in a straight line.
- **Nothing shields gravity.** No material placed between two bodies changes the pull between them.
- **Why the mango falls, not the Earth.** The mango and the Earth pull each other with the same force $F = GMm/r^2$. But the mango's acceleration is $F/m = GM/r^2 = g$, and the Earth's is $a = F/M = Gm/r^2$. So

$$\frac{a}{g} = \frac{m}{M}$$

For a 0.3 kg mango, $a/g = 0.3/(6 \times 10^{24}) = 5 \times 10^{-26}$: the Earth does move towards the mango, but far too little ever to notice.

Check 2

A planet has the same mass as the Earth but half its radius. The value of g on its surface is:

a g/2 b g/4
c 2g d 4g

7.4 Inertial and gravitational mass

The word "mass" is used for two things that are, on the face of it, different.

- **Inertial mass** is how much a body resists a change in its velocity. It is measured by pushing: apply a known force F, measure the acceleration a, and $m = F/a$ (Newton's second law). It could be measured in deep space, far from any planet.
- **Gravitational mass** is how strongly a body is pulled by gravity (and pulls). It is measured by weighing: in a field where the pull on it is F at distance d from a mass M, $m = \dfrac{Fd^2}{GM}$. A beam balance compares gravitational masses.

Experiments find the two **exactly equal** for every body: only the method of measuring differs. That is why all bodies fall with the same g: a body with twice the gravitational mass is pulled twice as hard, but has twice the inertial mass to move, so its acceleration is the same.

DID YOU KNOW?

Tested to one part in a million billion

In 1889 Loránd Eötvös showed with a torsion balance that inertial and gravitational mass agree to about one part in 20 million, and with his colleagues went on to test many materials, from wood to platinum. In 2022 the French satellite MICROSCOPE, comparing two cylinders, one of platinum and one of titanium, falling round the Earth for months, found no difference down to about one part in 10^{15}. Albert Einstein made this equality the starting point of his theory of gravity (general relativity, 1915).

7.5 Gravitational field strength

The **gravitational field** of a body is the region round it in which another mass feels its gravitational pull. In principle it reaches to infinity, getting weaker with distance.

The **gravitational field strength** (or **intensity**) E at a point is the **force on a unit mass** placed at that point:

$$E = \frac{F}{m}$$

It is a **vector**, pointing the way the force on the mass points. Its SI unit is N/kg (which is the same as m/s²). The test mass is taken small enough not to disturb the field it measures.

The field of a point mass, or a sphere

Place a test mass m at a distance r from the center of a body of mass M (a point mass, or a uniform sphere with r outside it).

1. By Newton's law, $F = \dfrac{GMm}{r^2}$.
2. The field strength is the force per unit mass:

$$E = \frac{F}{m} = \frac{GM}{r^2}$$

directed towards the center of M.

At the Earth's surface, $r = R$:

$$E = \frac{GM}{R^2} = g$$

So **the gravitational field strength at a point equals the acceleration due to gravity there**: 9.8 N/kg is the same as 9.8 m/s². A 1 kg mass feels a pull of 9.8 N, and falls with 9.8 m/s².

Check 19

10. What is the value of g at the center of the Earth?

a half its value at the surface b double its value at the surface
c equal to its value at the surface d zero

Check 20

11. What is the period of a satellite in a parking (geostationary) orbit?

The textbook's options are 1 year, 365 days, 8760 hours and "all of the above", and none of them fits: a parking orbit takes 24 hours, so the book's options don't work. The options below are corrected.

a 1 year b 24 hours
c 12 hours d 27.3 days

Check 21

12. For a body in a falling lift to be weightless, the lift's acceleration must be:

a equal to g b greater than g
c less than g d double g

Check 22

13. What is the force on a body due to the Earth called?

a gravitation b gravity
c reaction force d none of the above

Check 23

14. Where is the value of g greatest?

a on the surface of the Earth b below the Earth's surface
c above the Earth's surface d midway between the Earth and the Moon

Check 24

15. The value of g at the center of the Earth is:

a infinite b zero
c greater than at the equator d less than at the equator, but not zero

19. The gravitational potential at a point is the work done in bringing a body of unit mass from infinity to that point, without acceleration. On the basis of this:
    a. What happens when the unit mass is replaced by a body of a larger mass?
    b. How does gravitational potential differ from gravitational potential energy?
    c. How is the potential at the Earth's surface related to the acceleration due to gravity there?

20. a. What are the center of mass and the center of gravity of a body?
    b. A uniform rod 200 cm long, of mass 500 g, is balanced on a wedge at its 40 cm mark. A 2 kg mass hangs from the rod at the 20 cm mark, and an unknown mass m at the 160 cm mark, as in Figure 7.12. Find m for the rod to be in equilibrium. (g = 10 m/s²)

Exercise

Choose the correct option

Check 10

1. What happens to the period of a simple pendulum as it is taken higher above the Earth's surface?

a it increases
b it decreases
c it remains constant
d it first increases, then decreases

Check 11

2. What is the value of g on a planet with 1/80 of the Earth's mass and half its radius? (g on the Earth = 9.8 m/s²)

a 0.05 m/s²
b 0.5 m/s²
c 5.0 m/s²
d 5.2 m/s²

Check 12

3. If the Earth kept its mass but its radius shrank by 1%, the value of g at its surface would:

a increase by about 2%
b decrease by 1%
c increase by 1%
d decrease by ½%

Check 13

4. The orbital speed of a satellite of mass m is proportional to:

a m^0
b m
c $m^{1/2}$
d $m^{-1/2}$

Check 14

5. At what height above the Earth's surface is g half its value at the surface? (Radius of the Earth 6400 km)

a 3050 km
b 2650 km
c 3250 km
d 3800 km

Check 15

6. How deep inside the Earth (radius R) must a person go for their weight to be a quarter of its value at the surface?

a R/4
b 3R/4
c 5R/4
d 5R/2

Check 16

7. If the distance between the Earth and the Sun were half what it is, how many days would there be in a year?

a 129
b 730
c 182.5
d 64.5

Check 17

8. What is the period of a geostationary satellite?

a 12 hours
b 24 hours
c 18 hours
d 6 hours

Check 18

9. A missile misses its target and goes into a circular orbit of radius 4 times the radius of the geostationary orbit. What is its period?

a 4 days
b 8 days
c 12 days
d 18 days

Answer these questions

16. a. Why is G called the universal gravitational constant?

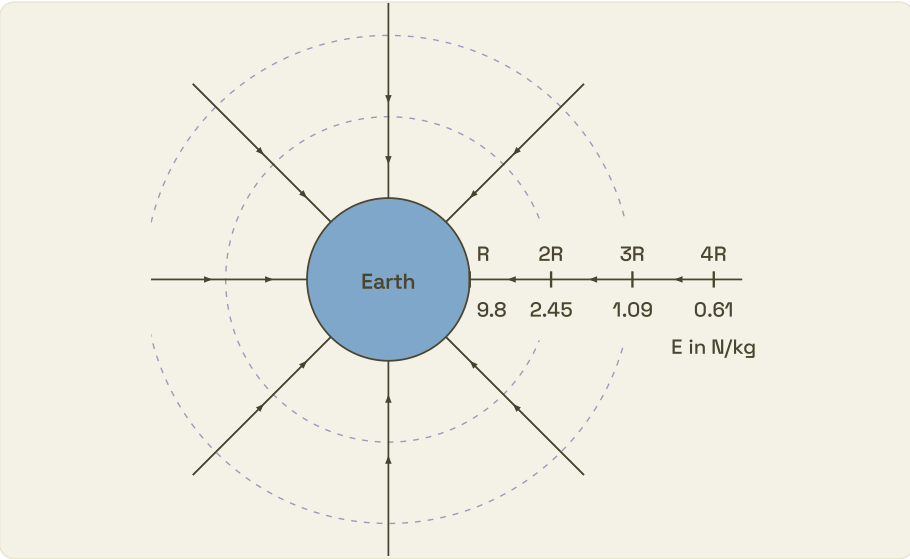


Figure 7.2 *The Earth's gravitational field lines all point to its center, crowded together near the surface where the field is strongest. Along one line, E at R, 2R, 3R and 4R from the center is 9.8, 2.45, 1.09 and 0.61 N/kg.*

WORKED EXAMPLE

The field at twice the Earth's radius

Find the gravitational field strength of the Earth at a height of 6.37×10^6 m above its surface. (Mass of the Earth $M = 5.98 \times 10^{24}$ kg, radius 6.37×10^6 m, $G = 6.67 \times 10^{-11}$ N m²/kg²)

Answer

The height equals the Earth's radius, so the point is $r = 2 \times 6.37 \times 10^6 = 1.274 \times 10^7$ m from the center.

$$E = \frac{GM}{r^2} = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(1.274 \times 10^7)^2} = \frac{3.989 \times 10^{14}}{1.623 \times 10^{14}} = 2.46 \text{ N/kg}$$

A quarter of the surface value, 9.83 N/kg, as the inverse square law says.

Force and field strength

- The **gravitational field strength** at a point is a property of the place: the pull on each kilogram there.
- The **gravitational force** is the pull on a particular body: the field strength times the body's mass, $F = mE$. "Gravitational pull" is just an everyday name for this force.

The field is different round different bodies because $E = GM/r^2$ depends on the mass that makes it. At its surface the Moon, with 1/81 of the Earth's mass and 0.27 of its radius, has a field of only 1.6 N/kg, a sixth of the Earth's. The Moon stays in orbit because it moves in the Earth's field.

Field strength from the potential

The field and the gravitational potential V (next section) are linked: **the field strength is minus the slope (gradient) of the graph of potential against distance**:

$$E = -\frac{dV}{dr}$$

The potential $V = -GM/r$ rises (becomes less negative) as r grows, so dV/dr is positive, and the minus sign makes the field point **inwards**, towards smaller r. Its size is $dV/dr = GM/r^2$, just as above. The steeper the potential graph, the stronger the field; far away the graph is nearly flat, and the field nearly zero.

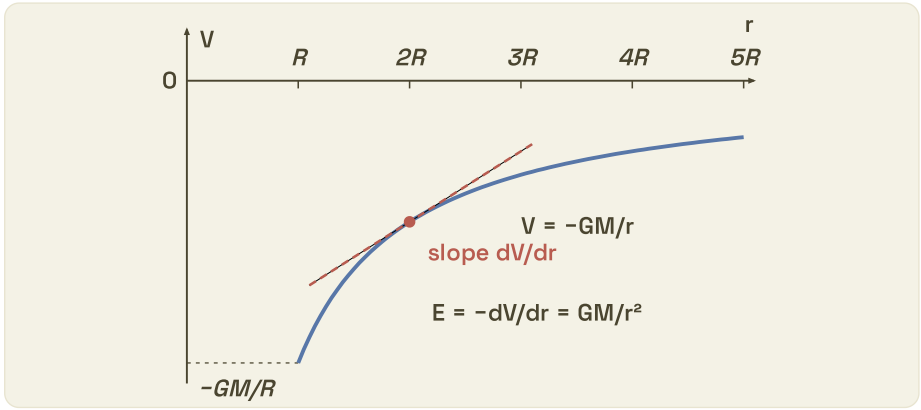


Figure 7.3 *The gravitational potential $V = -GM/r$ outside the Earth. It is negative everywhere and rises towards 0 far away. Its slope at any point (the dashed tangent, drawn at 2R) gives the field there: $E = -dV/dr$.*

Check 3

At a point the gravitational field strength is 4 N/kg. What force acts on a 3 kg mass there?

a 0.75 N
b 1.33 N
c 12 N
d 39.2 N

7.6 Gravitational potential

The **gravitational potential** V at a point in a gravitational field is the **work done in bringing a unit mass from infinity to that point**, without accelerating it. Its SI unit is the **joule per kilogram** (J/kg). It is a **scalar**.

"Infinity" is a place so far away that the pull of the body is zero there; we take the potential there to be zero.

Derivation: $V = -GM/r$

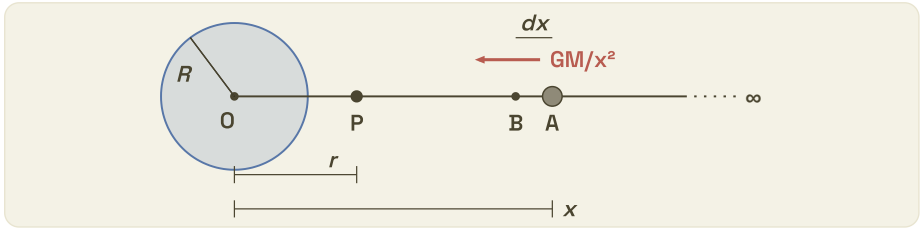


Figure 7.4 *A unit mass is brought from infinity to P, a distance r from the center O of the Earth. On the way it is at A, a distance x from O, where the Earth pulls it inwards with GM/x^2 per kilogram; B is a short step dx nearer.*

Measure distances x outwards from O, the center of the Earth (mass M).

1. At A, a distance x from O, the Earth pulls a unit mass **inwards** with a force $\dfrac{GM}{x^2}$.
2. To bring the mass in **without accelerating it**, we hold it back with an equal force $\dfrac{GM}{x^2}$ pointing **outwards**, in the direction of increasing x.
3. As it moves from A to B, x changes by dx (a negative amount, since x is getting smaller). The work we do is force × displacement along the force:

$$dW = \frac{GM}{x^2}\,dx$$

4. Add up the work from infinity ($x = \infty$) to P ($x = r$):

$$W = \int_{\infty}^{r} \frac{GM}{x^2}\,dx = GM\left[-\frac{1}{x}\right]_{\infty}^{r} = GM\left(-\frac{1}{r} + \frac{1}{\infty}\right) = -\frac{GM}{r}$$

Project work

IN SHORT

Key ideas of Unit 7

- Newton's law of gravitation: $F = G\dfrac{m_1 m_2}{r^2}$, with $G = 6.67 \times 10^{-11}$ N m²/kg², the force between two 1 kg masses 1 m apart.
- $g = GM/R^2$, independent of the falling body's mass; it lets us weigh the Earth, $M = gR^2/G$.
- Inertial mass ($m = F/a$) and gravitational mass (from the pull of gravity) are equal.
- Field strength $E = F/m = GM/r^2$, equal to g; $E = -dV/dr$.
- Potential $V = -GM/r$ (J/kg); potential energy $U = -GMm/r = mV$; zero at infinity, negative everywhere else.
- g varies: above the surface $g' = g\dfrac{R^2}{(R+h)^2} \approx g(1 - 2h/R)$; below, $g' = g(1 - d/R)$, zero at the center; less at the equator (bulge and spin) than at the poles.
- Center of mass $X = \dfrac{\sum m_i x_i}{\sum m_i}$; the center of gravity is where the resultant weight acts. Stability: low C.G., wide base, C.G. over the base.
- A satellite: $v = \sqrt{GM/r}$, $T = 2\pi\sqrt{r^3/GM}$ ($T^2 \propto r^3$); near the surface 7.9 km/s and about 85 minutes.
- Escape velocity $v_e = \sqrt{2GM/R} = \sqrt{2gR} = 11.2$ km/s for the Earth.
- A satellite's energies: $K = \dfrac{GMm}{2r}$, $U = -\dfrac{GMm}{r}$, $E = -\dfrac{GMm}{2r}$, negative because it is bound.
- A geostationary satellite: period one day, over the equator, west to east, about 36,000 km up, 3.1 km/s.
- Weightlessness comes from free fall, not from the absence of gravity.
- GPS finds a place by trilateration: distances from the signals' travel time at the speed of light; four satellites for position and the receiver's clock.

meet, one of which is usually far out in space or deep in the Earth, so the receiver picks the other. Finding a place from distances is called **trilateration** (finding it from **angles** would be triangulation, which GPS does not use).

Why four satellites. The satellites have atomic clocks, but a phone has an ordinary quartz clock, which can easily be thousandths of a second out. So every distance the receiver works out is wrong by the same amount, and three spheres no longer meet at one point. The receiver uses a **fourth** satellite to work out its own clock error as well: four unknowns (three for its position, one for its clock), four distances. So **a receiver needs signals from at least four satellites**, and uses all it can see (often 8 to 12) for a better fix. Its clock then keeps time as well as the satellites' atomic clocks.

The three segments

1. **The space segment:** the satellites, each sending its signal.
2. **The control segment:** a master control station (in Colorado, with a backup), about 16 monitoring stations round the world, and ground antennas. They track the satellites, work out their exact orbits and clock errors, and upload corrections to them.
3. **The user segment:** the GPS **receivers** in phones, cars, planes and ships, which pick up the signals and work out their position. They only receive; they send nothing back to the satellites.

What limits its accuracy

- **The atmosphere:** the signals slow down slightly in the ionosphere (charged gas 80–1,000 km up, disturbed by solar storms) and in the lower air.
- **Obstructions and reflections:** steep hillsides, tall buildings and trees block satellites or bounce their signals, so a signal arrives by a longer path. In a deep gorge or a narrow Kathmandu street the fix is worse.
- **Satellite geometry:** satellites bunched in one part of the sky give a poor fix; spread round the sky, a good one.
- **Orbit and clock errors** in the data the satellites send (the **ephemeris**, their predicted positions), and the receiver's own electronics.

A phone usually finds its place to within about 5 m; survey equipment, by comparing signals with a base station, to within a centimeter.

GPS and GNSS

GPS is run by the United States. It is one of several systems, together called **GNSS** (Global Navigation Satellite Systems): Russia's **GLONASS**, the European Union's **Galileo** and China's **BeiDou** also cover the whole world, and India's **NavIC** and Japan's **QZSS** cover their own regions (NavIC covers Nepal). Most phones today receive several systems at once: more satellites in view give a quicker and more accurate fix.

DID YOU KNOW?

Einstein in your pocket

The GPS satellites' clocks run fast by about 38 microseconds a day compared with clocks on the ground: they are higher in the Earth's field, which by Einstein's general relativity makes clocks tick faster (+45 μs a day), and they move fast, which by special relativity makes them tick slower (−7 μs a day). Uncorrected, that would put a fix out by about 10 km a day. So the satellites' clocks are set before launch to tick slightly slow: a piece of Einstein's physics every phone depends on.

DID YOU KNOW?

Measuring Sagarmatha

In 2019 Nepali surveyors carried a GNSS receiver to the summit of Sagarmatha. Together with gravity measurements round the mountain (needed to find where "sea level" lies under the Himalaya) and Chinese surveyors' work from the north, it gave the height announced in 2020: **8,848.86 m**.

Check 9

What is the least number of GPS satellites a receiver needs signals from to find its position on the Earth and its time?

a 2 b 3
c 4 d 24

5. This work, per unit mass, is the potential at P:

$$\boxed{V = -\frac{GM}{r}}$$

At the Earth's surface, with $g = 9.8$ m/s² and $R = 6.4 \times 10^6$ m, $V = -\dfrac{GM}{R} = -gR \approx -62.7$ MJ/kg.

Why is it negative? Gravity attracts, so on the way in the field pulls the mass along and we only hold it back: we do negative work. The potential is greatest, zero, at infinity, and less (more negative) everywhere closer. To lift a kilogram from the Earth's surface right out of its field, you must **give** it 62.7 MJ.

7.7 Gravitational potential energy

Near the surface, where g hardly changes, lifting a mass m by a height h stores mgh of potential energy: you met this in earlier grades. But when the height is comparable with the Earth's radius, g changes along the way, and we need the general form.

The **gravitational potential energy** of a body at a point in a gravitational field is the **work done in bringing the body from infinity to that point**, without accelerating it.

Derivation: $U = -GMm/r$

Repeat the derivation of 7.6 for a body of mass m instead of a unit mass (Figure 7.4).

1. At a distance x from the center, the pull on the body is $\dfrac{GMm}{x^2}$, inwards; we hold it back with the same force outwards.
2. Moving it by dx, we do work $dW = \dfrac{GMm}{x^2}\,dx$.
3. From infinity to P, at distance r:

$$W = \int_\infty^r \frac{GMm}{x^2}\,dx = GMm\left[-\frac{1}{x}\right]_\infty^r = -\frac{GMm}{r}$$

$$\boxed{U = -\frac{GMm}{r} = mV}$$

The potential energy of a body is its mass times the potential at its position. On the Earth's surface:

$$U_R = -\frac{GMm}{R}$$

It is negative for the same reason: the field does the work of bringing the body in. Only **changes** in potential energy are measured, and the zero is chosen at infinity.

WORKED EXAMPLE

Two stars falling together

Two stars, each of mass 2×10^{30} kg (the Sun's mass), are approaching each other. They start 10^9 km apart, with negligible speed. Each has a radius of 10^4 km. With what speed do they collide? ($G = 6.67 \times 10^{-11}$ N m²/kg²)

Answer

$M = 2 \times 10^{30}$ kg; starting distance $r = 10^9$ km $= 10^{12}$ m; radius $R = 10^4$ km $= 10^7$ m. They touch when their centers are $2R$ apart.

The only force is their pull on each other, so energy is conserved. By symmetry each reaches the same speed v.

- At the start: $U + K = -\dfrac{GMM}{r} + 0$
- When they touch: $U + K = -\dfrac{GMM}{2R} + \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 = -\dfrac{GM^2}{2R} + Mv^2$

Setting the two equal and dividing by M:

$$v^2 = GM\left(\frac{1}{2R} - \frac{1}{r}\right) = 6.67 \times 10^{-11} \times 2 \times 10^{30}\left(\frac{1}{2 \times 10^7} - \frac{1}{10^{12}}\right)$$

$$v^2 = 1.334 \times 10^{20} \times (5 \times 10^{-8} - 1 \times 10^{-12}) = 6.67 \times 10^{12} \quad \Rightarrow \quad v = 2.58 \times 10^6 \text{ m/s}$$

Each star hits at about 2,600 km/s, nearly 1% of the speed of light.

Check 4

A 2 kg body is at a point where the gravitational potential is −40 MJ/kg. Its gravitational potential energy is:

a −20 MJ
b −80 MJ
c +80 MJ
d −40 MJ

7.8 Variation of the acceleration due to gravity

7.8.1 Variation of g with altitude

Take a body at a height h above the surface of the Earth (mass M, radius R). Its distance from the center is $r = R + h$ (Figure 7.5).

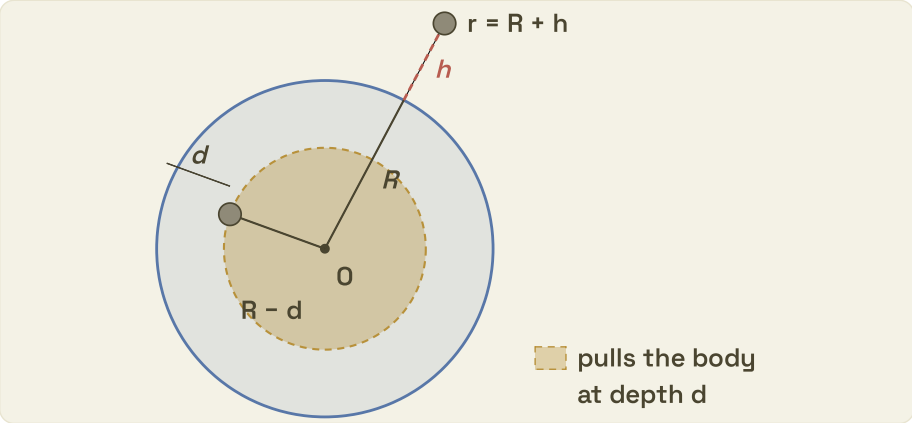


Figure 7.5 *A body at a height h above the surface, R + h from the center O, and one at a depth d below it, R − d from O. Only the shaded ball beneath the deep one pulls it overall: the rock above it pulls equally in every direction. (Not to scale.)*

1. On the surface: $g = \dfrac{GM}{R^2}$ (i)
2. At the height h: $g' = \dfrac{GM}{(R+h)^2}$ (ii)
3. Dividing (ii) by (i):

$$\boxed{g' = g\,\frac{R^2}{(R+h)^2} = \frac{g}{\left(1+\dfrac{h}{R}\right)^2}}$$

Since $R + h > R$, $g' < g$: **g decreases with height**, falling as $1/r^2$.

For a height small compared with the Earth's radius ($h \ll R$), the binomial approximation $(1+x)^{-2} \approx 1 - 2x$ gives

$$g' \approx g\left(1 - \frac{2h}{R}\right)$$

WORKED EXAMPLE

On the summit of Everest

Find g on the summit of Sagarmatha (Everest), 8,849 m above sea level. (Take $g = 9.8$ m/s² at sea level and $R = 6400$ km.)

Answer

$$g' \approx g\left(1 - \frac{2h}{R}\right) = 9.8\left(1 - \frac{2 \times 8.849}{6400}\right) = 9.8 \times (1 - 0.00277) = 9.77 \text{ m/s}^2$$

About 0.28% less: a climber's weight drops by less than 2 N. What makes the climb so hard is the thin air, not the gravity.

7.8.2 Variation of g with depth

Go down a mine to a depth d. Two effects compete: you are **closer** to the center, which strengthens the pull, but the rock **above** you no longer pulls you inwards. A uniform spherical shell of rock pulls nothing inside it at all (its pulls from all sides cancel

1. **at the center of the Earth**, where the pull is zero (7.8.2);
2. **in free fall**, as in a falling lift or a diving aircraft;
3. **in an orbiting spacecraft**: the spacecraft and everything in it fall freely round the Earth together (7.9), so nothing presses on anything else;
4. **at the point between the Earth and the Moon where their pulls are equal and opposite** (about 346,000 km from the Earth's center), where a body at rest feels no resultant pull.

CAREFUL

Weightless is not gravity-free

In cases 2 and 3, gravity is still acting, and strongly. The International Space Station, 420 km up, is in a field of about 8.6 N/kg, nearly 90% of that at the ground; that pull is exactly what keeps it in its orbit. Its crew float because they are falling freely with it, not because gravity has gone. For the same reason a pendulum on the station does not swing: with nothing holding the bob up, it falls with the station and has no restoring pull. (It is not that $g = 0$ there.) Astronauts' water floats as a wobbling ball, and they sleep tied to the wall.



Figure 7.11 *Astronaut Jack Fischer on the International Space Station watches his drink float in a ball. He, the drink and the station all fall round the Earth together.*
Photo: NASA, Public domain

7.13 Global Positioning System (GPS)

When a map app shows a blue dot where you stand, a ride app finds you at your door, or a trekker's watch records a route over Thorong La, they are using satellites. The **Global Positioning System (GPS)** is a **global navigation satellite system**: it lets a receiver anywhere on or near the Earth find its position, its velocity and the exact time, in any weather, day and night. Its main uses:

- **finding a location**, where you are;
- **navigation**, the way from one place to another, by air, sea and road;
- **tracking** the movement of a vehicle, an animal or a person;
- **mapping and surveying** the land;
- **timing**: its clocks give the exact time to banks, power grids and phone networks.

It is also used in farming and rescue work, and it is in every smartphone.

How GPS works

The satellites. GPS has at least 24 satellites, in six orbital planes with four slots each, so that at least four are above the horizon from anywhere on the Earth; with spares, around 30 are usually in use. They orbit about **20,200 km** above the Earth, going round **twice a day**, at about 3.9 km/s (14,000 km/h). Each carries **atomic clocks** and sends out, all the time, a radio (microwave) signal that says which satellite it is, where it is, and the exact time the signal left.

Distance from time. A receiver picks up the signal and notes when it arrived. The signal travels at the speed of light, $c = 3 \times 10^8$ m/s, so

$$\text{distance to the satellite} = c \times (\text{time the signal took})$$

A signal from a satellite overhead takes about 0.07 s. An error of only one microsecond in the timing would put the distance out by 300 m.

Place from distances: trilateration. Knowing its distance from one satellite puts the receiver somewhere on a sphere round that satellite. A second distance narrows it to the circle where two spheres meet; a third, to the **two points** where three spheres

YOUR TURN

The space station's energy

The International Space Station, about 420,000 kg, orbits at $r = 6.79 \times 10^6$ m. Find its kinetic, potential and total energy. ($GM = 3.99 \times 10^{14}$ N m²/kg)

Answer

$\frac{GMm}{r} = \frac{3.99 \times 10^{14} \times 4.2 \times 10^5}{6.79 \times 10^6} = 2.47 \times 10^{13}$ J.

- $K = \frac{1}{2} \times 2.47 \times 10^{13} =$ **1.23×10^{13} J**
- $U =$ **-2.47×10^{13} J**
- $E =$ **-1.23×10^{13} J**: that much would set it free of the Earth.

7.12 Geostationary satellites

A satellite whose period is the same as the Earth's period of rotation is in a **synchronous orbit**. If its orbit is also circular, in the **plane of the equator**, and goes round **from west to east** (the way the Earth turns), it stays over the same point on the equator: seen from the ground it is at rest. It is a **geostationary satellite**, in a **geostationary** (or **parking**) **orbit**. Geostationary satellites carry television, telephone and internet links and watch the weather: one dish, pointed once, receives a satellite all the time.

Its height. With $T = 24$ h $= 86{,}400$ s, $g = 9.8$ m/s² and $R = 6.4 \times 10^6$ m, from 7.9.2:

$$h = \left(\frac{gR^2T^2}{4\pi^2}\right)^{1/3} - R = \left(\frac{9.8 \times (6.4 \times 10^6)^2 \times (86{,}400)^2}{4\pi^2}\right)^{1/3} - 6.4 \times 10^6$$

$$h = (7.59 \times 10^{22})^{1/3} - 6.4 \times 10^6 = 4.23 \times 10^7 - 0.64 \times 10^7 = 3.59 \times 10^7 \text{ m} \approx 36{,}000 \text{ km}$$

Its speed:

$$v = \frac{2\pi r}{T} = \frac{2\pi \times 4.23 \times 10^7}{86{,}400} = 3.08 \times 10^3 \text{ m/s} \approx 3.1 \text{ km/s}$$

(The Earth really turns once in 23 h 56 min, its **sidereal day**, relative to the stars; with that, the exact height is 35,786 km.)

DID YOU KNOW?

Launching eastwards, near the equator

The Earth turns from west to east, carrying everything at its equator eastwards at 465 m/s (about 410 m/s at Nepal's latitude). A rocket launched **eastwards** starts with that speed for free, so it needs less fuel to reach orbital speed; and from near the **equator** it gains the most and is already in the plane of the geostationary orbit. That is why so many launch sites are near the equator: Europe launches from French Guiana, India from Sriharikota at 13.7° N.

Weightlessness and free fall

A body is in **free fall** when gravity is the **only** force acting on it. A body feels **weightless** when nothing pushes up on it, no floor, seat or scale: what we feel as our weight is really the push of the floor on us.

Free fall is the **cause**, weightlessness the **effect**. In a lift whose cable has snapped, you and the lift fall together with acceleration g: the floor no longer presses on you, a scale under you reads zero, and you float.

WORKED EXAMPLE

A falling lift

A 60 kg student stands on a scale in a lift. Find the scale's reading (her apparent weight) when the lift accelerates a. down at 3 m/s², b. up at 3 m/s², c. down at g. ($g = 9.8$ m/s²)

Answer

The forces on her are her weight mg down and the scale's push N up; N is what the scale reads.

a. Down at 3 m/s²: $mg - N = ma \Rightarrow N = m(g - a) = 60 \times (9.8 - 3) =$ **408 N** (her true weight is 588 N).

b. Up at 3 m/s²: $N - mg = ma \Rightarrow N = m(g + a) = 60 \times 12.8 =$ **768 N**.

c. Down at g: $N = m(g - g) =$ **0**: she is weightless.

A body feels weightless:

exactly), so only the ball of Earth beneath you, of radius $R - d$, counts.

Take the Earth as a uniform sphere of density ρ. A body at depth d is $r = R - d$ from the center (Figure 7.5).

1. The mass of the ball of radius r beneath it: $M' = \frac{4}{3}\pi r^3 \rho$.
2. The field it feels:

$$g' = \frac{GM'}{r^2} = \frac{G \cdot \frac{4}{3}\pi r^3 \rho}{r^2} = \frac{4}{3}\pi G \rho \, r$$

3. On the surface the same formula with $r = R$ gives $g = \frac{4}{3}\pi G \rho R$ (since $M = \frac{4}{3}\pi R^3 \rho$).
4. Dividing:

$$\boxed{g' = g\frac{r}{R} = g\left(1 - \frac{d}{R}\right)}$$

So inside a uniform Earth, g falls **in a straight line** with depth, to **zero at the center** ($d = R$). A body there has no weight: it is pulled equally in every direction.

Putting 7.8.1 and 7.8.2 together, g is greatest at the surface and less above and below it:

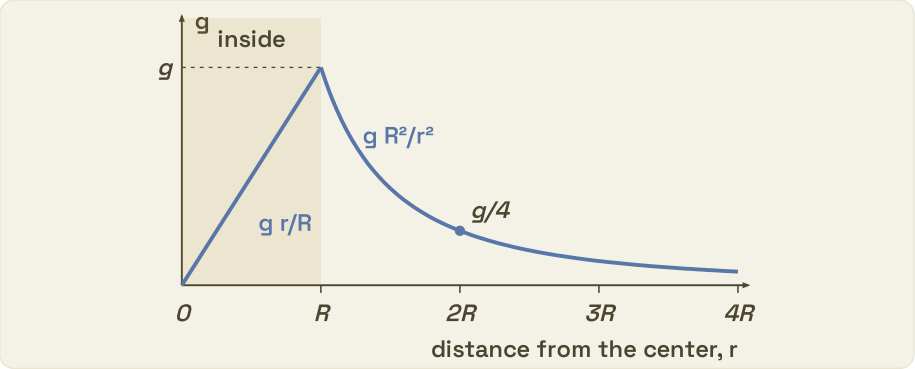


Figure 7.6 *g against distance from the Earth's center, for a uniform Earth. Inside, g rises in a straight line from 0 at the center to its surface value at R; outside it falls as $1/r^2$: g/4 at 2R, g/9 at 3R.*

DID YOU KNOW?

The real Earth: strongest at the core

The real Earth is not uniform: its iron core is about four times as dense as the surface rocks. Going down through the light crust and mantle, the loss of the rock above matters less than getting closer to the dense core, so g actually **rises** slightly with depth, to about 10.7 m/s² at the top of the core, 2,900 km down; only inside the core does it fall away to zero at the center. The straight-line graph above is for a uniform Earth, which is what the exam asks about.

Variation of g with latitude

The shape. The Earth bulges at the equator: its equatorial radius is 6,378 km and its polar radius 6,357 km. A body at a pole is closer to the center, so $g = GM/R^2$ is larger there.

The spin. A body on the equator is carried round a circle of radius 6,378 km once a day. Part of the Earth's pull is used up as the centripetal force for this circle, $mR\omega^2$, so less of it presses the body on the scale. With $\omega = 2\pi/86{,}164$ s $= 7.29 \times 10^{-5}$ rad/s:

$$R\omega^2 = 6.378 \times 10^6 \times (7.29 \times 10^{-5})^2 = 0.034 \text{ m/s}^2$$

At the poles, on the axis, the body goes round no circle, and the whole pull counts. The two effects together make g about 9.78 m/s² at the equator and 9.83 m/s² at the poles. (The bulge itself is caused by the spin, too: a soft spinning ball flattens.)

Check 5

At what depth below the surface does g become half its surface value (uniform Earth)?

a R/4 b R/2
c 3R/4 d R

7.9 Center of mass and center of gravity

7.9.1 Center of mass

The **center of mass** of a body is the point where its whole mass can be taken to be concentrated, for working out its motion as a whole. A force through the center of mass gives the body a linear acceleration with no turning; a force anywhere else turns it as well.

A body is made of many particles, of masses $m_1, m_2, \dots, m_n$, at positions $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

1. Let forces $F_1, F_2, \dots, F_n$ act on the particles, giving them accelerations $a_1 = \dfrac{d^2x_1}{dt^2}$, and so on. Adding Newton's second law for every particle (along x):
$$\sum F = m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2} + \cdots + m_n\frac{d^2x_n}{dt^2} = \frac{d^2}{dt^2}\left(m_1x_1 + m_2x_2 + \cdots + m_nx_n\right)$$
2. Write $M = m_1 + m_2 + \cdots + m_n$ for the total mass, and divide and multiply by it:
$$\sum F = M\,\frac{d^2}{dt^2}\left(\frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n}\right) = M\,\frac{d^2X}{dt^2}$$
3. So the whole body moves as if its total mass M were at the single point (X, Y) and all the forces acted there:
$$X = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n}, \qquad Y = \frac{m_1y_1 + m_2y_2 + \cdots + m_ny_n}{m_1 + m_2 + \cdots + m_n}$$

That point is the center of mass: the **mass-weighted average** of the particles' positions.

Two bodies. For two masses m_1 and m_2 a distance d apart, the center of mass lies on the line joining them, at distances r_1 from m_1 and r_2 from m_2 such that

$$m_1r_1 = m_2r_2, \qquad r_1 + r_2 = d \quad\Rightarrow\quad r_1 = \frac{m_2\,d}{m_1 + m_2}, \quad r_2 = \frac{m_1\,d}{m_1 + m_2}$$

It is **nearer the heavier mass**: a seesaw with a heavy and a light child balances closer to the heavy one.

WORKED EXAMPLE

Two particles

Two particles of masses 1 kg and 2 kg are at $x = 0$ and $x = 3$ m. Find the position of their center of mass.

Answer

From the formula: $X = \dfrac{1\times 0 + 2\times 3}{1+2} = \dfrac{6}{3} = \mathbf{2\ m}$ from the origin.
Or from $m_1r_1 = m_2r_2$: with the center of mass x from the 1 kg particle, $1\times x = 2\times(3-x)$, so $3x = 6$ and $x = 2$ m.
It is twice as far from the 1 kg particle as from the 2 kg one.

DID YOU KNOW?

The Earth wobbles too

The Earth and the Moon both go round their common center of mass every 27.3 days. The Earth is 81 times heavier, so that point is 81 times closer to the Earth's center than to the Moon's: $384{,}400/82 \approx 4{,}700$ km from the Earth's center, inside the Earth, 1,700 km below the surface. Astronomers find planets round other stars the same way: the star wobbles round the star–planet center of mass, and the wobble shows up in its light.

Center of gravity

The **center of gravity** (C.G.) of a body is the point through which the **resultant of the weights** of all its particles acts, vertically downwards, in whatever position the body is held. It is the average position of the body's weight.

- Where the gravitational field is uniform across the body (any ordinary object on the Earth), the center of gravity is at the **same point** as the center of mass. They differ only for a body so large that g changes across it (a tall mountain, or a long satellite pointing at the Earth).
- The C.G. **need not be inside the material** of the body: for a ring, a hollow ball or a horseshoe it is at an empty point.
- Builders and designers of buildings, bridges, ships, aircraft and kites must know where it is: a structure is most stable with its C.G. low and over the middle of its base.

7.11 Potential energy and kinetic energy of a satellite

A satellite going round a planet has **kinetic energy**, because it moves, and **potential energy**, because it is in the planet's field.

Take a satellite of mass m in a circular orbit of radius $r = R + h$ round the Earth (mass M).

1. The pull gives the centripetal force: $\dfrac{mv^2}{r} = \dfrac{GMm}{r^2}$, so $mv^2 = \dfrac{GMm}{r}$.
2. **Kinetic energy:**
$$K = \tfrac{1}{2}mv^2 = \frac{GMm}{2r}$$
3. **Potential energy** (7.7):
$$U = -\frac{GMm}{r}$$
4. **Total energy:**
$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

So, for any circular orbit,

$$\boxed{E = -\frac{GMm}{2r} = \frac{U}{2} = -K}$$

- The kinetic energy is positive and the potential energy negative, twice as large.
- The **total energy is negative**: that is what it means for the satellite to be **bound** to the Earth. To set it free (to bring its total energy up to zero, at rest at infinity), an amount $\dfrac{GMm}{2r}$ must be supplied from outside, for instance by its rocket. This is the satellite's **binding energy**.
- A higher orbit (larger r) has **more** total energy (less negative) but **less** kinetic energy: to raise a satellite, its rocket must add energy, yet it ends up going slower.

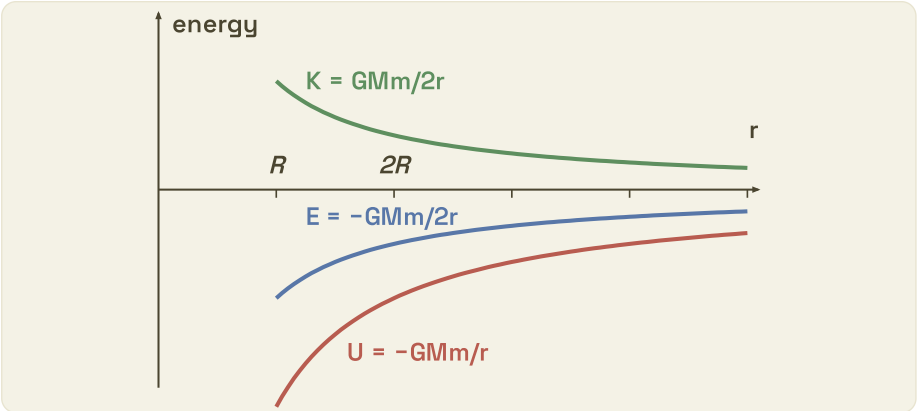


Figure 7.10 *A satellite's energies against the radius of its circular orbit. The kinetic energy K is positive, the potential energy U negative, and the total $E = K + U = -GMm/2r$ is negative: the satellite is bound. All three tend to 0 for very large orbits.*

7.10 Escape velocity

Throw a stone up and it comes back. Throw it faster and it goes higher before it returns. Since gravity weakens with height, there is a speed at which the stone climbs forever, slowing all the time but never stopping and never coming back.

The **escape velocity** is the **least velocity** with which a body must be projected from the surface of a planet so that it escapes the planet's gravitational pull and never returns (ignoring air resistance).

Derivation: $v_e = \sqrt{2gR}$

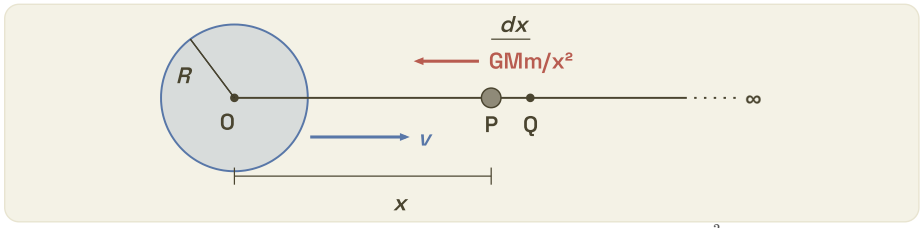


Figure 7.9 *A body leaves the Earth's surface at speed v. At P, a distance x from the center O, the Earth pulls it back with GMm/x^2; Q is a short step dx farther out. To escape, it must reach infinity.*

1. At P, a distance x from the center, the Earth pulls the body (mass m) back with $F = \dfrac{GMm}{x^2}$.
2. As it moves out by dx, the work done **against** gravity is $dW = \dfrac{GMm}{x^2}\,dx$.
3. The total work against gravity from the surface ($x = R$) to infinity:

$$W = \int_R^\infty \frac{GMm}{x^2}\,dx = GMm\left[-\frac{1}{x}\right]_R^\infty = GMm\left(0 + \frac{1}{R}\right) = \frac{GMm}{R}$$

(This is just the rise in potential energy, from $-GMm/R$ to 0.)

4. This work comes out of the body's kinetic energy. It just escapes if its kinetic energy is used up exactly as it reaches infinity:

$$\tfrac{1}{2}mv_e^2 = \frac{GMm}{R}$$

5. So

$$\boxed{v_e = \sqrt{\frac{2GM}{R}} = \sqrt{2gR}}$$

using $GM = gR^2$. For the Earth:

$$v_e = \sqrt{2 \times 9.8 \times 6.4 \times 10^6} = 1.12 \times 10^4 \text{ m/s} = 11.2 \text{ km/s}$$

- Below this speed the body cannot escape: it stops and falls back (or goes into orbit, if it is going sideways).
- Like the orbital velocity, it **does not depend on the body's mass**, nor (ignoring air) on the direction it is launched in.
- Comparing with the speed of an orbit just above the surface: $v_e = \sqrt{2}\,v_o = 1.414 \times 7.9 = 11.2$ km/s.
- It is **different for every planet**, since it depends on M/R. On the Moon it is only 2.4 km/s, which is why the Moon has no air: its gas molecules, moving at hundreds of meters a second (some much faster), have slowly leaked away into space.

DID YOU KNOW?

Black holes

If a body is squeezed small enough, $\sqrt{2GM/R}$ would exceed the speed of light, 3×10^8 m/s, and nothing, not even light, could escape: a **black hole**. For the Earth that would mean squeezing it into a ball 1.8 cm across. In 2019 the Event Horizon Telescope photographed the shadow of a real one, 6.5 billion times the Sun's mass, at the center of the galaxy M87.

Check 8

A planet has twice the Earth's radius and the same density. Its escape velocity is:

a the same as the Earth's
b √2 times the Earth's
c 2 times the Earth's
d 4 times the Earth's

Conditions for stable equilibrium

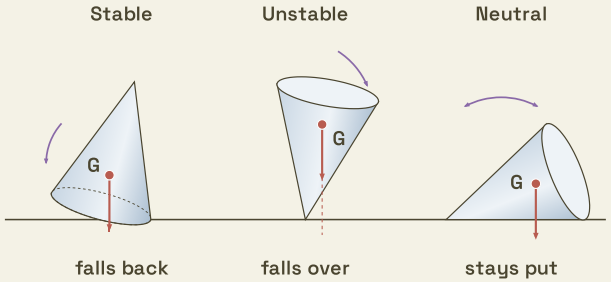


Figure 7.7 *A solid cone in the three kinds of equilibrium. Stable: tilted, its weight W still acts inside the base and turns it back. Unstable: on its tip, the weight acts outside the tiny base and turns it further over. Neutral: lying on its side, its C.G. stays at the same height as it rolls.*

A body stands most stably when:

1. its **center of gravity is as low as possible**;
2. its **base is as wide as possible**;
3. the vertical line through its C.G. falls **within its base**. If the line passes outside the base, the weight turns the body over.

Everyday examples:

- A **ship** carries heavy ballast (or its cargo) low in its hull, to keep its C.G. low.
- A **double-decker bus** with many passengers upstairs has its C.G. raised, so it tips over at a smaller tilt; such buses are tested on a tilt table with weights upstairs.
- Passengers who **stand up in a small boat** raise its C.G., and it capsizes more easily.
- A **racing car** is built wide and low.

Check 6

Why is it dangerous for passengers to stand up in a small boat?

a They make the boat heavier.
b They raise its center of gravity, so a smaller tilt can capsize it.
c They lower its center of gravity.
d They push the water away.

7.9 Motion of a satellite

A **satellite** is a body that orbits a larger body, held by its gravity. The Moon is the Earth's **natural satellite**; the Earth and the other planets are satellites of the Sun; and since 1957 people have launched thousands of **artificial satellites** round the Earth, for communication (telephone, TV, internet), weather forecasting, mapping, navigation, science and defense. In every case **the gravitational pull of the larger body provides the centripetal force** that keeps the satellite on its curved path.

How a satellite gets into orbit. Throw a stone horizontally from a high mountain: it falls in a curve (a projectile's path, Unit 3) and lands some distance away. Throw it faster, and it lands farther. The Earth is round, so the ground curves away beneath a fast enough stone; at the right speed the stone falls round the curve as fast as the ground drops away, and never lands: it is **in orbit**. This was Newton's own thought experiment. In practice a **rocket** carries a satellite above the air (which would slow it down) and gives it this sideways speed.

The first artificial satellite, **Sputnik 1**, was launched by the Soviet Union on 4 October 1957. Nepal's first satellite, **NepaliSat-1**, a cube of about 1 kg built by Nepali engineers in Japan, was launched in 2019 and released into orbit from the International Space Station.

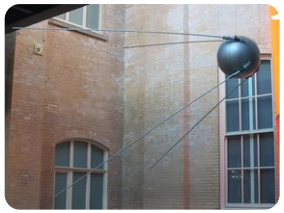


Figure 7.8 *A replica of Sputnik 1. The real one was a 58 cm metal ball weighing 84 kg, which went round the Earth every 96 minutes and sent out radio beeps for three weeks.*
Photo: Rept0n1x, CC BY-SA 3.0

7.9.1 Orbital velocity of a satellite

The **orbital velocity** of a satellite is the velocity with which it moves round its (circular) orbit.

Take a satellite of mass m moving in a circular orbit at a height h above the surface of the Earth (mass M, radius R), with orbital velocity v. The radius of its orbit is $r = R + h$.

1. The gravitational pull provides the centripetal force:
$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$
2. So $v^2 = \dfrac{GM}{r}$, and
$$\boxed{v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R+h}}}$$
3. Since $g = GM/R^2$ at the surface, $GM = gR^2$:
$$v = \sqrt{\frac{gR^2}{R+h}} = R\sqrt{\frac{g}{R+h}}$$

- v does **not depend on the satellite's mass**: a 1 kg cube and the 420,000 kg International Space Station at the same height go equally fast.
- The **higher** the orbit, the **slower** the satellite.

Close to the surface ($h \ll R$, an orbit just above the air):

$$v = \sqrt{gR} = \sqrt{9.8 \times 6.4 \times 10^6} = 7.92 \times 10^3 \text{ m/s} \approx 7.9 \text{ km/s}$$

About 28,500 km/h: from Kathmandu to Pokhara, about 140 km in a straight line, in under 20 seconds.

7.9.2 Time period of a satellite

The **time period** T of a satellite is the time it takes to go once round its orbit.

1. Its speed is $v = \dfrac{2\pi r}{T}$, so $T = \dfrac{2\pi r}{v}$.
2. With $v = \sqrt{GM/r}$:
$$\boxed{T = 2\pi\sqrt{\frac{r^3}{GM}} = 2\pi\sqrt{\frac{(R+h)^3}{gR^2}}}$$

(Equivalently, from $\dfrac{GMm}{r^2} = mr\omega^2$ and $\omega = 2\pi/T$.)

3. Squaring: $T^2 = \dfrac{4\pi^2}{GM} r^3$, so $T^2 \propto r^3$ for every satellite of the same planet. This is **Kepler's third law**, which Kepler found from the planets' orbits in 1619 and Newton explained.

Close to the surface, $r \approx R$:

$$T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{6.4 \times 10^6}{9.8}} = 2\pi \times 808 = 5077 \text{ s} \approx 1 \text{ h } 25 \text{ min}$$

The International Space Station, about 420 km up, takes about 92 minutes.

The height for a given period. Rearranging $T^2 = \dfrac{4\pi^2(R+h)^3}{gR^2}$:

$$(R+h)^3 = \frac{gR^2T^2}{4\pi^2} \quad \Rightarrow \quad h = \left(\frac{gR^2T^2}{4\pi^2}\right)^{1/3} - R$$

CAREFUL
Seasons are not about distance

The Earth's orbit round the Sun is an ellipse, very nearly a circle: its distance from the Sun changes by about 3% in a year, and the pull of the Sun by about 7%. But that is **not** what causes the seasons: the Earth is **nearest** the Sun in early January, in Nepal's winter. Seasons come from the tilt of the Earth's axis, which makes the Sun climb higher and stay up longer in summer.

DID YOU KNOW?
Why falling satellites burn up

A satellite in a low orbit brushes the thin top of the atmosphere, which slowly drags it lower. Once it reaches the thicker air below about 120 km, still moving at nearly 8 km/s, it squeezes the air in front of it so hard that the air heats to thousands of degrees, and most satellites burn up. A rocket going **up** does not, because it climbs slowly through the thick lower air and is moving fast only once the air is thin; a spacecraft coming back needs a heat shield.

WORKED EXAMPLE
g from the Moon's orbit

The Moon goes round the Earth in 27 days 8 hours, in an orbit of radius 60.1 times the Earth's radius. Find g at the Earth's surface. (Radius of the Earth $R = 6.36 \times 10^6$ m.)

Answer
$T = 27 \times 86{,}400 + 8 \times 3600 = 2{,}332{,}800 + 28{,}800 = 2{,}361{,}600$ s; $r = 60.1R$.
For the Moon, the pull gives the centripetal force:

$$\frac{GM}{r^2} = r\left(\frac{2\pi}{T}\right)^2 \quad \Rightarrow \quad GM = \frac{4\pi^2 r^3}{T^2}$$

With $GM = gR^2$:

$$g = \frac{4\pi^2 r^3}{R^2T^2} = \frac{4\pi^2(60.1)^3 R}{T^2} = \frac{39.48 \times 217{,}081 \times 6.36 \times 10^6}{(2.3616 \times 10^6)^2} = \frac{5.451 \times 10^{13}}{5.577 \times 10^{12}} = 9.77 \text{ m/s}^2$$

Newton did this calculation: the Moon's motion and a falling apple give the same g, proof that one law rules both.

WORKED EXAMPLE
A communication satellite

A proposed communication satellite is to orbit the Earth in a circle in the plane of the equator, 35,880 km above the surface. Find its period in hours, and comment. (Radius of the Earth 6370 km, mass of the Earth 5.98×10^{24} kg, $G = 6.67 \times 10^{-11}$ N m²/kg²)

Answer
$r = R + h = 6370 + 35{,}880 = 42{,}250$ km $= 4.225 \times 10^7$ m; $GM = 6.67 \times 10^{-11} \times 5.98 \times 10^{24} = 3.989 \times 10^{14}$ N m²/kg.

$$T = 2\pi\sqrt{\frac{r^3}{GM}} = 2\pi\sqrt{\frac{(4.225 \times 10^7)^3}{3.989 \times 10^{14}}} = 2\pi\sqrt{1.891 \times 10^8} = 2\pi \times 13{,}750 = 86{,}400 \text{ s} = 24.0 \text{ h}$$

Comment: its period equals the time the Earth takes to turn once. Going round over the equator, eastwards like the Earth, it keeps pace with the ground below and seems to stand still in the sky: a geostationary satellite (7.12).

Check 7
Satellite A orbits at a greater height than satellite B. Compared with B, A has:

a a greater speed and a shorter period
b a smaller speed and a longer period
c the same speed and a longer period
d a greater speed and a longer period