Unit 7

Answers: Gravitation

Quick checks

1 c $F \propto \dfrac{m_1 m_2}{r^2}$: doubling a mass doubles F; halving r multiplies it by 4. Together, $2 \times 4 = 8$ times.

2 d $g = GM/R^2$: with the same mass and half the radius, g is $1/(1/2)^2 = 4$ times larger.

3 c $F = mE = 3 \times 4 = 12$ N. The 9.8 belongs to the Earth's surface only; here the field is 4 N/kg.

4 b $U = mV = 2 \times (-40) = -80$ MJ. It would take 80 MJ to carry the body away to infinity.

5 b $g' = g(1 - d/R) = g/2$ gives $d = R/2$, about 3,200 km down.

6 b Standing up raises the C.G. of the boat and its load. A small tilt then takes the line of the weight beyond the point where the water's upthrust can turn the boat back.

7 b $v = \sqrt{GM/r}$ falls as r grows, and $T = 2\pi\sqrt{r^3/GM}$ grows. A higher satellite goes slower **and** has farther to go.

8 c $v_e = \sqrt{2GM/R}$ with $M = \frac{4}{3}\pi R^3 \rho$: $v_e = R\sqrt{\frac{8}{3}\pi G\rho} \propto R$ for the same density. Twice the radius, twice the escape velocity.

9 c Three distances fix a place in space only if the receiver's clock is perfect. The fourth satellite lets it solve for its clock error too: four unknowns, four distances.

10 a $T = 2\pi\sqrt{l/g}$, and g decreases with altitude, so T increases: a pendulum clock carried up a mountain runs slow.

11 b $g \propto M/R^2$: $g_p = 9.8 \times \dfrac{1/80}{(1/2)^2} = 9.8 \times \dfrac{4}{80} = 0.49 \approx 0.5$ m/s².

12 a $g \propto 1/R^2$. With $R' = 0.99R$: $g' = g/0.99^2 = 1.0203g$, about 2% more. (For small changes, the percentage change in g is −2 × the percentage change in R.)

13 a $v = \sqrt{GM/r}$: the satellite's own mass cancels, so $v \propto m^0$: it does not depend on m at all.

14 b $\dfrac{g}{2} = g\dfrac{R^2}{(R+h)^2} \Rightarrow (R+h)^2 = 2R^2 \Rightarrow R + h = \sqrt{2}R$.
$h = (\sqrt{2} - 1)R = 0.414 \times 6400 = 2651$ km ≈ 2650 km.

15 b $g' = g(1 - d/R) = g/4 \Rightarrow d/R = 3/4$, so $d = 3R/4$.

16 a By Kepler's third law, $T \propto r^{3/2}$: $T' = 365 \times (1/2)^{3/2} = 365/2.83 = 129$ days.

17 b It must go round in the time the Earth turns once: 24 hours (more exactly, one sidereal day, 23 h 56 min).

18 b $T \propto r^{3/2}$: $T = 1 \times 4^{3/2} = 1 \times 8 = 8$ days.

19 d $g' = g(1 - d/R)$; at the center $d = R$, so $g' = 0$: the Earth pulls equally in all directions.

20 b A parking orbit is the geostationary orbit, so its period is that of the Earth's rotation, 24 hours. (One year would be the Earth's orbit round the Sun; 27.3 days the Moon's orbit round the Earth.)

21 a The scale reads $N = m(g - a)$, which is zero when $a = g$: the lift and the body fall freely together.

22 b The pull of the **Earth** on a body is gravity; gravitation is the general attraction between any two bodies.

23 **a** For a uniform Earth, g is greatest at the surface and less above it ($\propto 1/r^2$) and below it ($\propto r$). (In the real Earth, with its dense core, g rises slightly with depth to about 10.7 m/s² at the core, but the uniform Earth is what this question means.)

24 **b** At the center the pulls of the surrounding matter cancel in every direction, so $g = 0$.

16. a. Its value, 6.67×10^{-11} N m²/kg², is the same for every pair of bodies, whatever they are made of, wherever they are in the universe, and whatever lies between them. It is the force between two 1 kg masses 1 m apart.

b. A body of mass m on the surface is pulled by the Earth with $F = \dfrac{GMm}{R^2}$ (Newton's law, with the Earth's mass taken at its center). By Newton's second law this force gives it the acceleration g: $F = mg$. So $mg = \dfrac{GMm}{R^2}$, and $g = \dfrac{GM}{R^2}$.

c. Its acceleration is the field strength at $r = 2R$: $a = \dfrac{GM}{(2R)^2} = \dfrac{1}{4} \cdot \dfrac{GM}{R^2} = \dfrac{g}{4} = \mathbf{2.45\ m/s^2}$, towards the center of the Earth.

17. a. The escape velocity is the least velocity with which a body must be projected from a planet's surface to escape its gravitational pull and never come back (ignoring air resistance).

b. To reach infinity, the body must do work against gravity equal to the rise in its potential energy, from $-GMm/R$ to 0:

$$W = \int_R^\infty \frac{GMm}{x^2}\,dx = \frac{GMm}{R}$$

Its kinetic energy must just provide this: $\frac{1}{2}mv_e^2 = \dfrac{GMm}{R}$, so $v_e = \sqrt{\dfrac{2GM}{R}} = \sqrt{2gR}$. With $g = 9.8$ m/s² and $R = 6.4 \times 10^6$ m, $v_e = \sqrt{2 \times 9.8 \times 6.4 \times 10^6} = \mathbf{11.2\ km/s}$.

c. $v_e = \sqrt{2GM/R}$ depends on the planet's mass and radius, which differ from planet to planet: 2.4 km/s on the Moon, 5.0 km/s on Mars, 60 km/s on Jupiter.

18. a. Weightlessness is the state in which nothing supports a body, so it feels no weight and a scale under it reads zero. It happens when the body is in free fall, with gravity the only force on it (a falling lift, an orbiting spacecraft), or where the resultant gravitational pull is zero.

b. The scale's push N and the weight mg: $mg - N = ma \Rightarrow N = m(g - a) = 50 \times (10 - 5) = \mathbf{250\ N}$.

c. $N - mg = ma \Rightarrow N = m(g + a) = 50 \times (10 + 5) = \mathbf{750\ N}$.

19. a. The potential at the point does not change: it is a property of the place, $V = -GM/r$, the work per kilogram. The work done on the larger body, its potential energy $U = mV$, is larger in proportion to its mass.

b. Potential is the work per unit mass (J/kg), a property of a point in the field; potential energy is the work for a particular body (J), its mass times the potential, $U = mV$.

c. At the surface $V = -\dfrac{GM}{R}$, and $GM = gR^2$, so $V = -gR$ (about −62.7 MJ/kg, with $g = 9.8$ m/s² and $R = 6400$ km).

20. a. The **center of mass** is the point where the whole mass of a body can be taken to be concentrated: a force through it moves the body without turning it, and its position is $X = \dfrac{\sum m_i x_i}{\sum m_i}$. The **center of gravity** is the point through which the resultant weight of the body acts, in every position of the body. In a uniform field (any ordinary object near the Earth) the two are at the same point.

b. Take moments about the wedge (40 cm). The rod's weight acts at its middle, the 100 cm mark.

- Anticlockwise: the 2 kg mass, 0.20 m from the wedge: $2 \times 10 \times 0.20 = 4$ N m.
- Clockwise: the rod's weight, 0.60 m away: $0.5 \times 10 \times 0.60 = 3$ N m; and m, 1.20 m away: $m \times 10 \times 1.20 = 12m$ N m.

In equilibrium the moments balance: $4 = 3 + 12m$, so $m = \dfrac{1}{12}$ kg $\approx$ **0.083 kg** (83 g).

21. a. Each satellite sends a signal saying where it is and the exact time it was sent. The receiver measures how long each signal took to arrive and multiplies by the speed of light to get its distance from each satellite. Each distance places it on a sphere round that satellite; the point where the spheres meet is its position (**trilateration**).

b. **Four**: three distances for the three coordinates of its position, and a fourth because the receiver's clock is not exact, which lets it work out its clock error as well.

c. Russia's GLONASS, the European Union's Galileo and China's BeiDou (worldwide); India's NavIC and Japan's QZSS (regional).

22. a. The acceleration due to gravity is the acceleration of a body falling freely under the Earth's gravity alone. On the surface $g = GM/R^2$; at a height h, $g' = \dfrac{GM}{(R+h)^2}$. Dividing:

$$g' = g\frac{R^2}{(R+h)^2}$$

which is less than g, so g decreases with height (for $h \ll R$, $g' \approx g(1 - 2h/R)$).

b. Reduced by 36%, $g' = 0.64g$:

$$\frac{R^2}{(R+h)^2} = 0.64 \Rightarrow \frac{R}{R+h} = 0.8 \Rightarrow R + h = 1.25R \Rightarrow h = 0.25R = 0.25 \times 6400 = 1600$$

(The approximation $1 - 2h/R$ would give 1152 km: it is only good for small heights.)

23. a. **Zero**: at the center $g = 0$, because the Earth's matter pulls equally in every direction.

b. For a uniform Earth of density ρ, at depth d only the sphere of radius $r = R - d$ beneath pulls: $g' = \dfrac{G \cdot \frac{4}{3}\pi r^3 \rho}{r^2} = \frac{4}{3}\pi G \rho r$. At the surface $g = \frac{4}{3}\pi G \rho R$. So $g' = g\dfrac{R-d}{R} = g\left(1 - \dfrac{d}{R}\right)$: g falls in proportion to the depth, to zero at the center.

c. $r = 6400 + 600 = 7000$ km.

$$F = mg\frac{R^2}{r^2} = 75 \times 9.8 \times \left(\frac{6400}{7000}\right)^2 = 735 \times 0.836 = 614$$

d. On the surface the force is $75 \times 9.8 = 735$ N. The ratio is $\dfrac{614}{735} = 0.84$, about **5 : 6**: the astronaut is still pulled with five-sixths of the weight felt on the ground, yet floats, because the astronaut and the telescope are falling freely round the Earth.

24.

a. The body must do work GMm/R against gravity all the way to infinity, all from the kinetic energy it was launched with. For the Earth this is about 62.7 MJ for every kilogram, which needs a speed of 11.2 km/s.

b. As in question 17b: $\frac{1}{2}mv_e^2 = \dfrac{GMm}{R} \Rightarrow v_e = \sqrt{\dfrac{2GM}{R}} = \sqrt{2gR}$.

c. $v_e = \sqrt{2GM/R}$, so

$$\frac{v_M}{v_E} = \sqrt{\frac{M_M}{M_E} \cdot \frac{R_E}{R_M}} = \sqrt{\frac{1}{9} \times 2} = \sqrt{\frac{2}{9}} = 0.471$$

$v_M = 0.471 \times 11.2 = \mathbf{5.3\ km/s}$.

25.

a. It goes round over the equator, from west to east, in the same time as the Earth turns once, so it stays above the same point on the equator. Seen from the ground it does not move ("geostationary": stationary relative to the Earth).

b. The pull gives the centripetal force: $\dfrac{GMm}{r^2} = mr\left(\dfrac{2\pi}{T}\right)^2$, so $r^3 = \dfrac{GMT^2}{4\pi^2} = \dfrac{gR^2T^2}{4\pi^2}$ (using $GM = gR^2$). With $r = R + h$:

$$h = \left(\frac{gR^2T^2}{4\pi^2}\right)^{1/3} - R$$

With $T = 86{,}400$ s this gives about 36,000 km.

c. $g = 980\ \text{cm/s}^2 = 9.8\ \text{m/s}^2$; $r = 6400 + 300 = 6700\ \text{km} = 6.7 \times 10^6\ \text{m}$.

$$v = R\sqrt{\frac{g}{r}} = 6.4 \times 10^6 \sqrt{\frac{9.8}{6.7 \times 10^6}} = 6.4 \times 10^6 \times 1.209 \times 10^{-3} = 7.74 \times 10^3 = 7.74$$

$$T = \frac{2\pi r}{v} = \frac{2\pi \times 6.7 \times 10^6}{7.74 \times 10^3} = 5439 \approx 131$$

26. a.

Gravitational potential	Gravitational potential energy
Work done in bringing a **unit mass** from infinity to the point	Work done in bringing a **given body** from infinity to the point
A property of the point in the field	Belongs to the body (and the body that makes the field)
$V = -GM/r$	$U = -GMm/r = mV$
Unit J/kg	Unit J

b. Bring the body (mass m) in from infinity to the surface, holding it back against the pull GMm/x^2 at each distance x. The work done is

$$W = \int_{\infty}^{R} \frac{GMm}{x^2}\,dx = GMm\left[-\frac{1}{x}\right]_{\infty}^{R} = -\frac{GMm}{R}$$

so $U_R = -\dfrac{GMm}{R}$.

c. They touch when their centers are $2R = 2 \times 10^6$ m apart. Energy is conserved, and each reaches the same speed v:

$$-\frac{GM^2}{r} = -\frac{GM^2}{2R} + 2 \cdot \tfrac{1}{2}Mv^2 \Rightarrow v^2 = GM\left(\frac{1}{2R} - \frac{1}{r}\right)$$

$$v^2 = 6.67 \times 10^{-11} \times 2 \times 10^{25} \times \left(\frac{1}{2 \times 10^6} - \frac{1}{10^{12}}\right) = 1.334 \times 10^{15} \times 5.0 \times 10^{-7} = 6.67 \times 10^8$$

$v = \mathbf{2.58 \times 10^4\ m/s}$ each.

27. a. The speed with which a satellite moves round its orbit (for a circular orbit, constant, along the tangent).

b. $\dfrac{GMm}{r^2} = \dfrac{mv^2}{r} \Rightarrow v = \sqrt{\dfrac{GM}{r}}$; $T = \dfrac{2\pi r}{v} = 2\pi\sqrt{\dfrac{r^3}{GM}}$, with $r = R + h$.

c. First find the orbit's radius: $r = \dfrac{GM}{v^2} = \dfrac{3.989 \times 10^{14}}{(6200)^2} = 1.038 \times 10^7$ m (about 4,000 km up).

i. $T = \dfrac{2\pi r}{v} = \dfrac{2\pi \times 1.038 \times 10^7}{6200} = 1.052 \times 10^4$ s $\approx$ **175 min**.

ii. $a = \dfrac{v^2}{r} = \dfrac{(6200)^2}{1.038 \times 10^7} = \mathbf{3.70\ m/s^2}$, towards the Earth's center (the field strength at that height).

28. $r = 6.4 \times 10^6 + 3.6 \times 10^6 = 1.0 \times 10^7$ m; $GM = 6.7 \times 10^{-11} \times 6 \times 10^{24} = 4.02 \times 10^{14}$.

$$T = 2\pi\sqrt{\frac{r^3}{GM}} = 2\pi\sqrt{\frac{10^{21}}{4.02 \times 10^{14}}} = 2\pi\sqrt{2.488 \times 10^6} = 2\pi \times 1577 = 9.91 \times 10^3$$

About **9910 s**, 2 h 45 min.

29. $g = \dfrac{GM}{R^2}$ with $M = \frac{4}{3}\pi R^3\rho$ gives $g = \frac{4}{3}\pi G\rho R$:

$$g = \tfrac{4}{3} \times 3.1416 \times 6.7 \times 10^{-11} \times 5500 \times 6.4 \times 10^6 = 9.88^2 \approx 9.9^2$$

30. The center of mass is $r_1 = \dfrac{m_2 d}{m_1 + m_2} = \dfrac{2}{3}d$ from the lighter star. For the lighter star, the pull gives its centripetal force:

$$\frac{Gm_1m_2}{d^2} = m_1 r_1 \omega^2 = m_1 \frac{m_2 d}{m_1 + m_2}\omega^2 \Rightarrow \omega^2 = \frac{G(m_1+m_2)}{d^3}$$

$$\omega^2 = \frac{6.7 \times 10^{-11} \times 3 \times 10^{20}}{(10^6)^3} = 2.01 \times 10^{-8} \Rightarrow \omega = 1.42 \times 10^{-4}$$

(The heavier star gives the same ω, as it must: they go round together.) That is one turn every 12 hours.

31. $r = 6370 + 189 = 6559$ km $= 6.559 \times 10^6$ m. The field strength is the centripetal acceleration: $\dfrac{v^2}{r} = 9.4$.

i. $v = \sqrt{9.4 \times 6.559 \times 10^6} = \sqrt{6.166 \times 10^7} =$ **7852 m/s**.

ii. $T = \dfrac{2\pi r}{v} = \dfrac{2\pi \times 6.559 \times 10^6}{7852} =$ **5249 s** (87.5 min).

32. At the surface the field is 10 N/kg; at $r = \frac{3}{2}R$ it is $10 \times \left(\dfrac{R}{1.5R}\right)^2 = 10 \times \dfrac{4}{9} = 4.44$ N/kg.

$F = 200 \times 4.44 =$ **889 N**.

33. $GM = 6.670 \times 10^{-11} \times 5.957 \times 10^{24} = 3.973 \times 10^{14}$ N m²/kg.

$$g_{pole} = \frac{3.973 \times 10^{14}}{(6.357 \times 10^6)^2} = 9.8323^2, \qquad g_{eq} = \frac{3.973 \times 10^{14}}{(6.378 \times 10^6)^2} = 9.7676^2$$

$\Delta g = 9.8323 - 9.7676 =$ **0.0646 m/s²** (carry four decimal places: the difference is small).
Other factors: the Earth's **rotation** (at the equator $R\omega^2 = 0.034$ m/s² of the pull goes into the circular motion; at the poles none), the **uneven distribution of mass** (the extra mass of the equatorial bulge, mountains, dense rocks), and the **height** of the place above sea level.

34. As in the worked example of 7.9: $T = 2.3616 \times 10^6$ s, and

$$g = \frac{4\pi^2 (60.1)^3 R}{T^2} = \frac{39.48 \times 2.171 \times 10^5 \times 6.36 \times 10^6}{(2.3616 \times 10^6)^2} = 9.77^2 \approx 9.8^2$$

35. $T = 2\pi\sqrt{\dfrac{r^3}{GM}}$ with $GM = gR^2$ and $r = 60R$:

$$T = 2\pi\sqrt{\frac{(60R)^3}{gR^2}} = 2\pi\sqrt{\frac{216{,}000\,R}{g}} = 2\pi\sqrt{\frac{216{,}000 \times 6.4 \times 10^6}{9.8}} = 2\pi \times 3.756 \times 10^5 = 2.36 \times 10^6$$

$T = \dfrac{2.36 \times 10^6}{8.6 \times 10^4} =$ **27.4 days**.

36.

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{1 \times 0 + 2 \times 3}{3} = 2$$

The center of mass is **2 m from the origin**, 1 m from the heavier particle.

37. $r = 6.4 \times 10^6 + 0.6 \times 10^6 = 7.0 \times 10^6$ m.

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{7.0 \times 10^6}} = \sqrt{5.698 \times 10^7} = 7.55 \times 10^3$$

About **7.55 km/s**, horizontally (along the orbit).