Answers: Circular Motion

Quick checks

1 b Once the string breaks, no force pulls the stone sideways, so it keeps the velocity it had at that instant: east, along the tangent (and it starts to fall, as any projectile does). The common belief that it flies straight outwards is wrong: nothing pushes it outwards.

2 c The only sideways force on the stone is the string's pull, towards her hand: that tension *is* the centripetal force. The centrifugal force is not a force on the stone seen from the ground; and the weight acts downwards, not towards the center of a horizontal circle.

3 c The resultant force is centripetal, always at 90° to the motion, so it does no work: $W = Fs\cos 90^\circ = 0$. That is why the speed stays at 20 km/h.

4 c 120 rpm = 2 rev/s, and each revolution is 2π rad, so ω = 2π × 2 = 4π ≈ 12.6 rad/s. 2 rad/s takes a revolution for a radian; 2π rad/s is one revolution a second, not two; 754 rad/s forgets to divide by 60.

5 b $v = r\omega = 0.35 \times 10 = 3.5$ m/s. 28.6 m/s divides instead of multiplying; 10 m/s confuses rad/s with m/s.

6 b $a = \dfrac{v^2}{r} = \dfrac{100}{50} = 2$ m/s², towards the center of the bend. 0.2 m/s² is v/r, the angular velocity (in rad/s), not an acceleration.

7 c $a = v^2/r$: doubling v multiplies v^2 by 4. This is why a car taking a bend twice as fast needs four times the grip.

8 c $F = \dfrac{mv^2}{r} = \dfrac{0.5 \times 16}{1} = 8$ N. 2 N forgets to square the speed; 16 N forgets the mass.

9 b $\tan\theta = v^2/(rg)$: a larger speed means a larger θ, so the bob swings outwards and rises, as the chair-swing seats do when the ride speeds up. It never quite reaches the horizontal.

10 b At the top the water needs $v^2/r \geq g$, so $v \geq \sqrt{gr} = \sqrt{9.8 \times 1} = 3.1$ m/s. 7.0 m/s is $\sqrt{5gr}$, the least speed needed at the **bottom** if the bucket moves freely.

11 a At the bottom the tension must both hold the body up against its weight and supply the centripetal force: $T = mv^2/r + mg$, and (moving freely) the speed is greatest there too.

12 b $\tan\theta = \dfrac{v^2}{rg} = \dfrac{400}{200 \times 9.8} = 0.204$, so θ = 11.5°. 78.5° is the angle with the vertical instead of the horizontal.

13 **b** $v_{\max} = \sqrt{\mu g r} = \sqrt{0.5 \times 10 \times 20} = \sqrt{100} = 10$ m/s, or 36 km/h. 100 m/s forgets the square root.

14 **a** The Moon goes round the Earth once in about 27.3 days on a path that is very nearly a circle (its distance varies by about 5%), at a nearly steady speed: approximately uniform circular motion, with the Earth's gravity as the centripetal force.

15 **c** The speed is constant, so the kinetic energy ½mv² doesn't change: the change is zero. The momentum does change: after ¾ of a turn the velocity has turned through 270°, at right angles to the start, so $|\Delta \vec{p}| = mv\sqrt{2}$ (and "mvr" isn't even a momentum: its unit is wrong).

16 **d** $v = 900 \div 3.6 = 250$ m/s. At the lowest point the centre is above, so the air's force F (up) minus the weight gives the centripetal force:

$$F - mg = \frac{mv^2}{r}$$

$$F = \frac{5000 \times 250^2}{2000} + 5000 \times 10$$

$$F = 156\,250 + 50\,000 = 2.0625 \times 10^5$$

upward.

17 **a** $T = 4\pi^2 m f^2 r$, so $f \propto \sqrt{T}$. Doubling T multiplies f by $\sqrt{2}$: $5 \times 1.414 = 7.07 \approx 7$ rpm.

18 **a** $f = 14 \div 25 = 0.56$ Hz, so $a = 4\pi^2 f^2 r = 4\pi^2 \times 0.56^2 \times 0.80 = 9.90$ m/s², towards the centre.

19 **a** $a = \dfrac{v^2}{r} = \dfrac{900}{500} = 1.8$ m/s², towards the centre.

20 **a** $a = r\omega^2 \Rightarrow \omega = \sqrt{\dfrac{a}{r}} = \sqrt{\dfrac{9.8}{0.20}} = \sqrt{49} = 7$ rad/s.

21 **a** $r = 6.4 \times 10^6$ m, $T = 24 \times 3600 = 86\,400$ s.

$$a = \frac{4\pi^2 r}{T^2} = \frac{39.48 \times 6.4 \times 10^6}{(86\,400)^2}$$

$$a = \frac{2.53 \times 10^8}{7.46 \times 10^9} = 0.034^2$$

That is about 0.3% of g, which is why a body weighs very slightly less at the equator.

22 **c** $F = mv^2/r \propto v^2$: twice the speed needs four times the force.

23 **b** $45^\circ = \dfrac{\pi}{4}$ rad $= 0.785$ rad. $r = \dfrac{s}{\theta} = \dfrac{0.22}{0.785} = 0.28$ m.

24 **d** The speed is constant, but the velocity keeps turning, and the acceleration, always towards the centre, keeps turning too. Their sizes stay the same; their directions change.

25 a Equal angles in equal times is uniform circular motion: the speed (magnitude) is constant and only the direction changes.

26 b $F = mr\omega^2 = 2 \times 1 \times 2^2 = 8$ N.

27 a $a = 4\pi^2 f^2 r = 4\pi^2 \times 20^2 \times 0.25 = 400\pi^2$ m/s² (about 3950 m/s²).

28 c It appears only when motion is described from a rotating frame; no body exerts it. It is a fictitious (pseudo) force.

29 b $\tan\theta = \dfrac{v^2}{rg} = \dfrac{65^2}{747 \times 9.8} = \dfrac{4225}{7321} = 0.577$, and $\tan 30^\circ = 0.577$: $\theta = 30°$.

30 a $a = r\omega^2 = r(2\pi f)^2 = 4\pi^2 r f^2$. (At a steady frequency its tangential acceleration is zero: the speed doesn't change.)

31 d Every part of the rolling ring turns about its center at the same angular velocity ω, so $F = m\omega^2 r \propto r$: the ratio is $\dfrac{m\omega^2 R_1}{m\omega^2 R_2} = \dfrac{R_1}{R_2}$.

19. a.

Linear motion	Circular motion
The body moves along a straight line.	The body moves along a circle, at a fixed distance from a center.
Described by displacement, velocity and acceleration along the line.	Described also by angular displacement, angular velocity and angular acceleration.
At a steady speed, the velocity is constant and no net force acts.	At a steady speed, the velocity still changes direction, so there is a centripetal acceleration and a net force towards the center.
Its direction can stay the same.	Its direction changes at every point.

b. One revolution takes T seconds, so in one second the body makes $1/T$ revolutions. The number of revolutions per second is the frequency, so $f = \dfrac{1}{T}$.

c. Its velocity, along the tangent, keeps changing direction, and a change of velocity is an acceleration (v^2/r, towards the center). By Newton's second law an acceleration needs a resultant force, mv^2/r towards the center. The force is at right angles to the motion, so it changes only the direction, never the speed.

20.

a. The **angle of banking** is the angle through which the outer edge of a road (or the outer rail of a track) at a bend is raised above the inner edge, the angle the road makes with the horizontal. It is needed because a vehicle on a bend needs a centripetal force: on a banked road the horizontal part of the normal reaction provides it, so the vehicle needn't rely on friction, which may be small on a wet or icy road and wears the tyres.

b. $v = 50 \div 3.6 = 13.9$ m/s.

$$\tan\theta = \frac{v^2}{rg} = \frac{13.9^2}{200 \times 9.8} = \frac{193}{1960} = 0.0984 \Rightarrow \theta = 5.6^\circ$$

21.

a. The horizontal part of the **tension** in the string, $T\sin\theta$, towards the center of the circle. (The vertical part, $T\cos\theta$, balances the weight.)

b.

Simple pendulum	Conical pendulum
The bob swings to and fro in a vertical plane along an arc.	The bob goes round a horizontal circle; the string sweeps out a cone.
Its speed changes: greatest at the middle, zero at the ends.	Its speed is constant.
Gravity provides the restoring force back to the middle.	The tension's horizontal part provides the centripetal force.
Period $2\pi\sqrt{l/g}$ (small swings).	Period $2\pi\sqrt{(l\cos\theta)/g}$.

c. $f = \dfrac{100}{60} = 1.667$ rev/s, so $\omega = 2\pi f = 2\pi \times 1.667 = \mathbf{10.47\ rad/s}$.

22.

a. Acceleration is the rate of change of **velocity**, and velocity has a direction. Moving in a circle, the body's direction changes all the time, so its velocity changes and it accelerates, towards the center, at v^2/r, even though its speed is constant.

b. In Bohr's model the electron is moving round the nucleus at high speed. The nucleus's attraction provides exactly the centripetal force it needs for its circular orbit, mv^2/r: the pull keeps bending its path into a circle instead of pulling it in, just as the Earth's gravity keeps the Moon in orbit without pulling it down. (Bohr also had to *postulate* that electrons in these orbits don't lose energy by radiating; quantum mechanics later explained why.)

c. $$F = \frac{4\pi^2 mr}{T^2} = \frac{4\pi^2 \times 0.2 \times 0.5}{0.5^2} = \frac{3.948}{0.25} = 15.8$$

23. a. No. Its **size**, v^2/r, is constant, but its **direction** keeps changing, always pointing to the center from wherever the particle is. A vector whose direction changes is not constant.

b. The **friction** between the tyres and the road, directed towards the center of the bend.

c. $a = \dfrac{v^2}{r}$ is the same for both, so $v^2 \propto r$: $\dfrac{v_1^2}{v_2^2} = \dfrac{r_1}{r_2} = \mathbf{2:3}$.

24. a. To turn, the plane needs a centripetal force. The lift on its wings acts at right angles to the wings. By banking (tilting its wings), the plane tilts the lift towards the center of the turn: the lift's vertical part holds the plane up, and its horizontal part provides the centripetal force, with $\tan\theta = v^2/(rg)$.

b. Circumference $= vT = 50 \times 40 = 2000$ m, so $r = \dfrac{2000}{2\pi} = 318$ m.

$$a = \frac{v^2}{r} = \frac{2500}{318} = 7.85^2$$

(Or directly $a = v\omega = v \times \dfrac{2\pi}{T} = 50 \times \dfrac{2\pi}{40} = 7.85$ m/s², towards the center.)

25. a. **Angular displacement** is the angle swept out at the center by the radius vector of a body moving round a circle, $\theta = s/r$, in radians.

b. As in Section 6.3: in a short time Δt the body goes from A to B and the radius turns through θ. The velocities $\vec{v}_A$, $\vec{v}_B$, each of size v, drawn from one point form a triangle PQR with $QR = |\Delta\vec{v}|$, similar to triangle OAB (both isosceles with angle θ between the equal sides). So $\dfrac{\Delta v}{v} = \dfrac{AB}{r}$, and with $AB \approx v\Delta t$, $\dfrac{\Delta v}{\Delta t} = \dfrac{v^2}{r}$. As $\Delta t \to 0$, $a = \dfrac{v^2}{r}$, directed towards the center. By Newton's second law the force is

$$F = ma = \frac{mv^2}{r}$$

towards the center.

c. $\omega_h = \dfrac{\omega_m}{12}$. The tips' speeds: $v_h = r_h\omega_h$, $v_m = r_m\omega_m$, with $v_h = \dfrac{v_m}{20}$:

$$r_h\frac{\omega_m}{12} = \frac{r_m\omega_m}{20} \Rightarrow r_h = \frac{12}{20}r_m = \frac{12}{20} \times 10 = 6$$

26.

a. A **conical pendulum** is a bob hanging on a string that moves round a horizontal circle at a constant speed, the string keeping a constant angle with the vertical and sweeping out a cone.

b. Forces on the bob: weight mg down, tension T along the string. Vertically $T\cos\theta = mg$; horizontally $T\sin\theta = mv^2/r$. Dividing, $\tan\theta = \dfrac{v^2}{rg}$. From the geometry $\tan\theta = \dfrac{r}{h}$, with $h = l\cos\theta$. So $\dfrac{r}{h} = \dfrac{v^2}{rg}$, giving $\omega^2 = \dfrac{v^2}{r^2} = \dfrac{g}{h}$. The period is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

c. i. $\dfrac{mv^2}{r} = \dfrac{10\times 25}{4} = 62.5$ N; $mg = 100$ N.

$$T = \sqrt{(mg)^2 + \left(\frac{mv^2}{r}\right)^2} = \sqrt{100^2 + 62.5^2} = \sqrt{13\,906} = 118$$

ii. $\tan\theta = \dfrac{v^2}{rg} = \dfrac{25}{40} = 0.625 \Rightarrow \theta = \mathbf{32°}$ to the vertical.

27.

a. **Banking** a track means raising its outer edge (or outer rail) above the inner one at a bend, so that the track slopes down towards the center of the bend.

b. On the banked road the normal reaction R is at θ to the vertical. Vertically $R\cos\theta = mg$; horizontally $R\sin\theta = mv^2/r$. Dividing, $\tan\theta = \dfrac{v^2}{rg}$, so $\theta = \tan^{-1}\dfrac{v^2}{rg}$.

c. Unbanked, the outer rail's sideways push on the wheel flanges provides all the centripetal force; the thrust of the wheels on the rail is equal and opposite:

$$F = \frac{mv^2}{r} = \frac{10^5\times 20^2}{150} = 2.67\times 10^5$$

For no thrust: $\tan\theta = \dfrac{v^2}{rg} = \dfrac{400}{150\times 9.8} = 0.272 \Rightarrow \theta = \mathbf{15.2°}$.

28.

a. **Motion in a vertical circle** is the motion of a body round a circle in a vertical plane (a stone whirled on a string beside you, a Ferris wheel, a roller-coaster loop). Gravity acts along the path, so the tension (or the normal force) changes round the circle, and, unless driven, the speed changes too.

b. At the top, the tension and the weight both point to the center: $T + mg = \dfrac{mv^2}{r}$. The string can't push, so $T \geq 0$; the least speed is when $T = 0$: $mg = \dfrac{mv^2}{r}$, so the **critical velocity** is $v = \sqrt{gr}$. (Moving freely, the body needs $\sqrt{5gr}$ at the bottom to arrive at the top with it.)

c. $v = 110 \div 3.6 = 30.6$ m/s. Friction must provide the centripetal force: $\dfrac{mv^2}{r} \leq \mu mg$, so

$$r \geq \frac{v^2}{\mu g} = \frac{30.6^2}{0.60 \times 9.8} = \frac{934}{5.88} = 159$$

The least radius is about **159 m**.

29.

It is a conical pendulum with $l = 0.40$ m, $r = 0.10$ m. $\sin\theta = \dfrac{0.10}{0.40} = 0.25$, so $\cos\theta = \sqrt{1 - 0.0625} = 0.968$ and $h = l\cos\theta = 0.387$ m.

$$\omega = \sqrt{\frac{g}{h}} = \sqrt{\frac{9.8}{0.387}} = \sqrt{25.3} = 5.03$$

30.

It turns once in 3600 s: $\omega = \dfrac{2\pi}{3600} = \mathbf{1.745 \times 10^{-3}\ rad/s}$.

In 20 minutes, a third of a turn: $\theta = \dfrac{2\pi}{3} = \mathbf{2.09\ rad}$ (120°).

$v = r\omega = 0.05 \times 1.745 \times 10^{-3} = \mathbf{8.73 \times 10^{-5}\ m/s}$ (under a tenth of a millimeter per second).

31.

$$T = \frac{mv^2}{r} \Rightarrow v = \sqrt{\frac{Tr}{m}} = \sqrt{\frac{600 \times 0.9}{0.8}} = \sqrt{675} = 26.0$$

32.

$$T = \frac{mg}{\cos\theta} = \frac{0.2 \times 9.8}{\cos 30^\circ} = \frac{1.96}{0.866} = 2.26$$

$$\tan\theta = \frac{v^2}{rg} \Rightarrow v = \sqrt{rg\tan\theta} = \sqrt{0.5 \times 9.8 \times 0.577} = \sqrt{2.83} = 1.68$$

Tension about **2.3 N**, speed about **1.7 m/s**.

33.

$v = 36 \div 3.6 = 10$ m/s. $\tan\theta = \dfrac{v^2}{rg} = \dfrac{100}{50 \times 9.8} = 0.204$, $\theta = 11.5°$.

The rise across the 1 m gauge: $h = d\sin\theta = 1 \times \sin 11.5^\circ = \mathbf{0.20\ m}$. (For small angles $\sin\theta \approx \tan\theta$, giving the same 0.20 m.)

34. $f = \dfrac{14}{25} = 0.56$ Hz, $\omega = 2\pi f = 3.52$ rad/s.
$a = r\omega^2 = 0.80 \times 3.52^2 = \mathbf{9.9\ m/s^2}$, directed along the radius **towards the center** of the circle.

35. The bob goes round with the car, a conical pendulum: $\tan\theta = \dfrac{v^2}{rg} = \dfrac{100}{10 \times 10} = 1$, so $\theta = \mathbf{45°}$, the bob swung out away from the center of the track.

36. This is the worked example in Section 6.6.

Where: at the lowest point, where the tension is greatest, 50 cm above the ground.

Angular speed: $20 = \dfrac{0.5\,v^2}{0.5} + 0.5 \times 10 \Rightarrow v = \sqrt{15} = 3.87$ m/s; $\omega = v/r = \mathbf{7.75\ rad/s}$.

Landing: fired horizontally from 0.50 m: $t = \sqrt{2 \times 0.5/10} = 0.316$ s; it lands $3.87 \times 0.316 = \mathbf{1.22\ m}$ from the point below the break, after 0.32 s in the air.

37.
$$a = 4\pi^2 f^2 r \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{a}{r}} = \frac{1}{2\pi}\sqrt{\frac{90}{5}} = \frac{4.24}{6.28} = 0.675$$

38.

i. The least tension is at the top: $T_{\min} = \dfrac{mv^2}{r} - mg \Rightarrow 3 = 0.4v^2 - 4$, so $v^2 = 17.5$, $v = \mathbf{4.2\ m/s}$.

ii. At the bottom: $T_{\max} = \dfrac{mv^2}{r} + mg = 7 + 4 = \mathbf{11\ N}$.

iii. With the string horizontal, the weight is along the tangent: $T = \dfrac{mv^2}{r} = \mathbf{7\ N}$.