Unit 4

Dynamics

This unit online, with its interactives:
science-playground.com/s/11f4

A loaded bus leaving Ratnapark lurches forward and every standing passenger sways back. A cricket fielder pulls her hands in as she catches a hard drive. A rocket at Sriharikota rises on a column of fire with nothing to push against. A tractor in Chitwan needs four people to start a stuck trailer moving, and two to keep it going.

In Unit 3 you learned to describe motion: where, how fast, which way. This unit asks **why** things move as they do. **Dynamics** is the branch of mechanics that studies motion together with its causes: forces. Its heart is three short laws written by Isaac Newton in 1687. With them, and with the ideas of momentum, impulse, torque and friction, you can explain every one of those scenes, and work out the numbers.

4.1 Introduction

People had argued about motion for two thousand years before it was put right. The Greek philosopher Aristotle taught that a moving body needs a force to keep it moving, and stops when the force stops. It seems obvious: a cart stops when the horse stops pulling.

In the early 1600s the Italian scientist **Galileo Galilei** (1564–1642) tested the idea. He rolled smooth balls down one ramp and up another and found that, the smoother the ramps, the closer the ball came to rising to its starting height, however gentle the second slope. If the second ramp were flat and perfectly smooth, he reasoned, the ball would roll on forever. So a force is needed not to keep a body moving but to **change** its motion: friction is the force that stops the cart. Galileo also worked out how bodies fall (Unit 3).

Isaac Newton (1643–1727) built on Galileo. In his *Principia* (1687) he stated the three laws of motion that are still the basis of mechanics for everything larger than an atom and slower than light: how a force changes a body's velocity, and how bodies push on each other.

4.1.1 Force

A **force** is a push or a pull that one body exerts on another, which changes, or tends to change, the body's state of rest or of uniform motion in a straight line (and can also change its shape). It is a vector: it has a size and a direction. Its SI unit is the **newton** (N), defined below by Newton's second law.



Figure 4.1 *Galileo Galilei, painted by Justus Sustermans in 1636. His ramps showed that a force changes motion, rather than keeping it going.*
Photo: Justus Sustermans, Public domain

4.1.2 Inertia

Every body tends to stay in the state it is in: at rest if it is at rest, moving steadily in a straight line if it is moving. It resists any change in its state of motion. This resistance is called **inertia**.

The **mass** of a body is the measure of its inertia: the greater the mass, the greater the inertia, and the harder it is to start, stop or turn the body.

- Two buckets stand in the yard, one full of water, one empty. A strong gust of wind topples or slides the empty one; the full one doesn't move. The full bucket has much more mass, so much more inertia.
- An iron ball and a hollow rubber ball of the same size roll towards you at the same speed. The rubber ball is far easier to stop: less mass, less inertia.

Inertia shows itself in three ways:

- **Inertia of rest.** A bus starts suddenly and the standing passengers sway backwards. Their feet move off with the floor, but their upper bodies tend to stay at rest. Shake a mango tree and ripe mangoes fall: the branch moves away, but the mango tends to stay where it was, and its weak stalk snaps.
- **Inertia of motion.** A moving bus brakes suddenly and the passengers lurch forwards. Their feet stop with the floor, but their upper bodies tend to keep moving.
- **Inertia of direction.** A bus turns sharply left and the passengers lean to the right: their bodies tend to keep moving in the straight line they were moving in, while the bus turns out from under them.

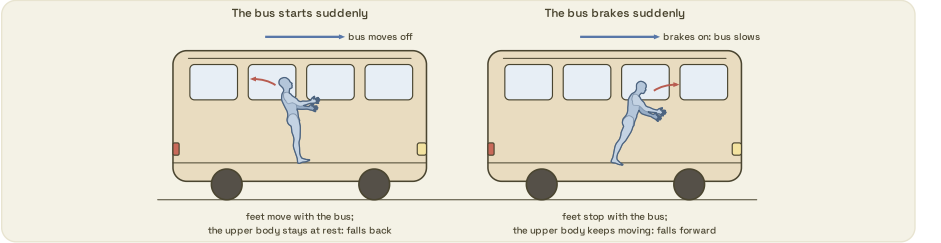


Figure 4.2 *Inertia in a bus. Left: the bus starts suddenly; the passenger's feet go with the floor, her upper body stays behind, and she falls back (inertia of rest). Right: the bus brakes; the feet stop, the upper body carries on, and she falls forward (inertia of motion).*

Check 1

A bus moving north turns sharply to the east. The standing passengers lurch towards the:

a east
b north-east, along the turn
c west (the outside of the turn)
d south

4.1.3 Linear momentum and impulse

A cyclist and a loaded truck both move at 5 m/s. Which is harder to stop? The truck, because it has far more mass. A rifle bullet weighs only a few grams but is dangerous because it is so fast. How hard a moving body is to stop depends on both its mass and its velocity, and their product is its **linear momentum**:

$$\vec{p} = m\vec{v}$$

Momentum is a vector in the direction of the velocity. Its SI unit is the kilogram meter per second, kg m/s (the same as N s). A body's momentum changes only if its velocity changes, that is, only if it accelerates, and Newton found that the **rate** at which a force changes a body's momentum is the force itself (Section 4.3.2).

WORKED EXAMPLE

Momentum of a bullet and a cyclist

Compare the momentum of a 10 g bullet at 400 m/s with that of a 60 kg cyclist at 5 m/s.

Answer

Bullet: $p = 0.010 \times 400 = \mathbf{4\ kg\ m/s}$. Cyclist: $p = 60 \times 5 = \mathbf{300\ kg\ m/s}$, 75 times as much. The bullet is dangerous not because of its momentum but because it delivers it to a tiny area in a tiny time (and because of its kinetic energy, Unit 5).

Impulse

Think of three things: a slap on the cheek, a slip on a wet road, two vehicles colliding. In each, a large force acts for a very short time, and the motion changes suddenly. The product of a force and the time for which it acts is the **impulse** of the force:

$$\text{impulse} = \vec{F}\,\Delta t$$

Impulse is a vector in the direction of the force; its unit is the newton second, N s, which is the same as kg m/s.

Newton's second law (Section 4.3.2) says that a force is the rate of change of momentum:

$$\vec{F} = \frac{m\vec{v} - m\vec{u}}{\Delta t} \quad \Rightarrow \quad \vec{F}\,\Delta t = m\vec{v} - m\vec{u}$$

The impulse of a force equals the change in momentum it produces. This is the **impulse–momentum theorem**. A running man who slips, a ball struck by a bat, two cars in a crash: in each, the impulse of the force is exactly the change in momentum.

When the force changes during the impact (as it nearly always does), F is the **average** force, and the impulse is the **area under the force–time graph**.

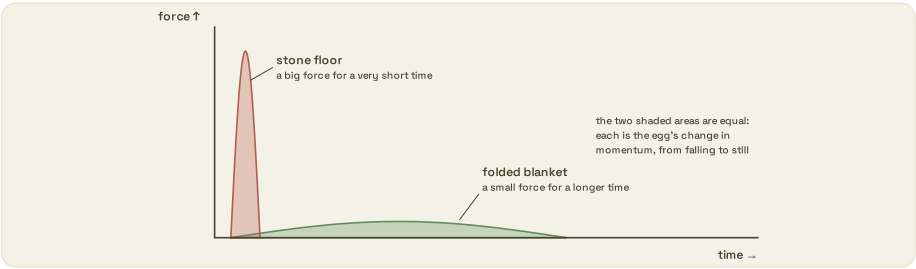


Figure 4.3 *The same egg dropped from the same height onto stone and onto a folded blanket. Both stop it, so both give it the same change in momentum: the areas under the two force–time curves are equal. The stone stops it in a short time with a large force; the blanket in a long time with a small force.*

The same change in momentum can be made by a large force for a short time or a small force for a long time. So, to **reduce the force** of an impact, **make it last longer**:

- A cricketer catching a hard ball draws her hands back with it, stretching the time the ball takes to stop, so the force on her hands is smaller.
- Falling on a rock or a concrete floor stops you almost at once: a huge force, and an injury. Falling on sand or a mattress stops you over a longer time with a smaller force.
- Glasses and china are packed in straw, paper or bubble wrap, so that a knock is spread over a longer time with a smaller force.
- Cars have crumple zones and airbags; helmets have foam liners; high jumpers land on thick mats.

The opposite is also used: a hammer delivers a large impulse in a very short time, so it drives a nail with a very large force.

WORKED EXAMPLE

A ball bounces off a wall

A ball of mass 0.500 kg moving at 15.0 m/s hits a wall and bounces straight back at 10.5 m/s. Find its initial and final momenta and the impulse on it. If the ball exerted an average force of 1000 N on the wall, how long did the collision last?

Answer

Take the direction **away from the wall** (the rebound) as positive. Then $u = -15.0$ m/s and $v = +10.5$ m/s.

Initial momentum: $mu = 0.500 \times (-15.0) =$ **−7.50 kg m/s** (towards the wall).

Final momentum: $mv = 0.500 \times 10.5 =$ **+5.25 kg m/s** (away from the wall).

Impulse = change in momentum $= mv - mu = 5.25 - (-7.50) =$ **12.75 kg m/s**, away from the wall. Notice the two momenta **add**: reversing a velocity changes the momentum by more than stopping it would.
Time: the wall pushes on the ball with 1000 N (equal and opposite to the ball's push, Section 4.3.3), so $F\,\Delta t = 12.75$ gives

$$\Delta t = \frac{12.75}{1000} = 0.0128 \text{ s} \approx 0.013 \text{ s}$$

Check 2

A 0.15 kg ball reaches a batter at 30 m/s and leaves the bat at 30 m/s straight back. The size of the impulse on the ball is:

a 0
b 4.5 N s
c 9 N s
d 18 N s

4.2 Principle of conservation of linear momentum

Two bodies collide: billiard balls, two carts, a bat and a ball. During the collision each pushes on the other, and each one's momentum changes. But what happens to the **total**?

Statement: for a system of two (or more) colliding bodies, **the total momentum before the collision equals the total momentum after it, provided no external force acts on the system**. More generally, the total momentum of an isolated system (one with no resultant external force on it) stays constant.

Derivation (from Newton's second and third laws). Two bodies of masses m_1 and m_2 move along one line with velocities u_1 and u_2, with $u_1 > u_2$, so the first catches up with the second (Figure 4.4). They are in contact for a short time Δt, and then

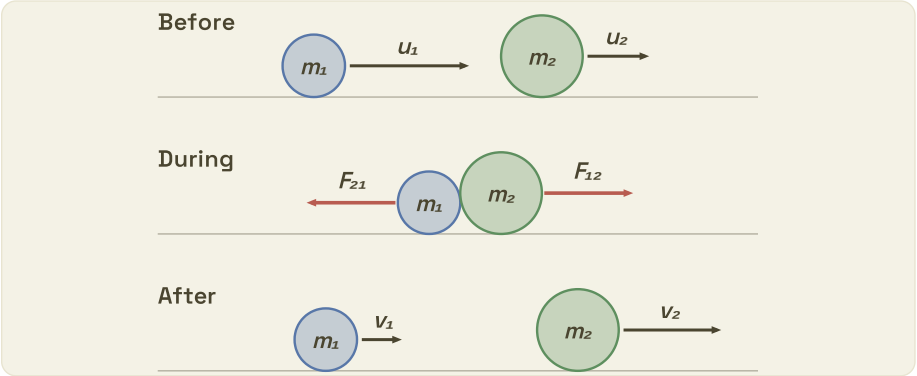


Figure 4.4 *A head-on collision in one line. Before: body 1 (mass m_1) moving at u_1 catches up with body 2 (m_2) moving at u_2, with $u_1 > u_2$. During: each pushes on the other, F_{12} on 2 and F_{21} on 1, equal and opposite. After: they separate at v_1 and v_2, with $v_2 > v_1$.*

move on with velocities v_1 and v_2.

1. During the contact, body 1 pushes body 2 with a force F_{12}. By the second law, this force is the rate of change of body 2's momentum:

$$F_{12} = \frac{m_2 v_2 - m_2 u_2}{\Delta t}$$

2. At the same time body 2 pushes body 1 with F_{21}, the rate of change of body 1's momentum:

$$F_{21} = \frac{m_1 v_1 - m_1 u_1}{\Delta t}$$

3. By the third law the two forces are equal and opposite, $F_{12} = -F_{21}$, and they act for the same time Δt:

$$\frac{m_2 v_2 - m_2 u_2}{\Delta t} = -\frac{m_1 v_1 - m_1 u_1}{\Delta t}$$

4. Multiply by Δt and rearrange:

$$\boxed{m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2}$$

Total momentum before = total momentum after. The forces between the two bodies are **internal** to the system: they change each body's momentum, but by equal and opposite amounts, so the total cannot change. Only an **external** force (friction from the ground, a push from outside) can change the total momentum.

The law holds whether the bodies bounce apart, stick together, or fly apart in an explosion, and in two or three dimensions (each component separately). It holds for every collision ever measured, from galaxies to subatomic particles.

Some examples:

- **A gun recoils.** Before firing, gun and bullet are at rest: total momentum zero. After, the bullet has forward momentum and the gun an equal backward momentum.
- **A rocket** pushes hot gas backward at high speed; the rocket gains an equal forward momentum, even in empty space with nothing to push against.
- **Jumping from a boat** onto the bank, you push the boat away from the bank.

WORKED EXAMPLE

Two balls collide

A ball A of mass 0.1 kg moving at 6 m/s hits a ball B of mass 0.2 kg at rest, head-on.

a. If they move off together, find their common velocity.

b. If instead A bounces straight back at 2 m/s, find B's velocity.

Answer

Take A's initial direction as positive: $m_A = 0.1$ kg, $u_A = 6$ m/s, $m_B = 0.2$ kg, $u_B = 0$.

a. Moving together at v: $m_A u_A + m_B u_B = (m_A + m_B)v$

$$v = \frac{0.1 \times 6 + 0.2 \times 0}{0.1 + 0.2} = \frac{0.6}{0.3} = 2 \text{ m/s}$$

2 m/s in A's original direction.

b. Now $v_A = -2$ m/s: $m_A u_A + m_B u_B = m_A v_A + m_B v_B$

$$0.1 \times 6 + 0 = 0.1 \times (-2) + 0.2\, v_B \quad \Rightarrow \quad 0.2\, v_B = 0.8 \quad \Rightarrow \quad v_B = 4 \text{ m/s}$$

4 m/s in A's original direction.

WORKED EXAMPLE

A ball drums on a wall

A ball of mass 0.05 kg hits a smooth wall at right angles 4 times in 2 seconds, each time at 10 m/s, bouncing back at 10 m/s. Find the average force on the wall.

Answer

Each bounce changes the ball's momentum from $+0.05 \times 10 = 0.5$ to -0.5 kg m/s: a change of 1.0 kg m/s. Four bounces change it by 4.0 kg m/s in 2 s, so the average force on the ball is

$$F = \frac{\text{total change in momentum}}{\text{time}} = \frac{4 \times 0.05 \times [10 - (-10)]}{2} = \frac{4.0}{2} = 2 \text{ N}$$

By the third law the ball pushes the wall with an average force of **2 N**. (This is how a gas presses on its container: countless molecules drumming on the walls, Unit 13.)

WORKED EXAMPLE

A toy rocket

A toy rocket of mass 2 kg, at rest on a smooth horizontal track, starts to throw out gas from its rear at 0.1 kg per second, at 50 m/s relative to the rocket. Find its initial acceleration.

Answer

Each second, 0.1 kg of gas is given a momentum of $0.1 \times 50 = 5$ kg m/s backward. The force pushing the gas back is the rate of change of its momentum:

$$F = \frac{\Delta m}{\Delta t} v = 0.1 \times 50 = 5 \text{ N}$$

By the third law, the gas pushes the rocket forward with 5 N (the **thrust**). The rocket's initial acceleration is

$$a = \frac{F}{m} = \frac{5}{2} = 2.5 \text{ m/s}^2$$

(Fired straight up, it would also have to beat its weight, about 20 N, so this rocket couldn't lift off vertically.)

Check 3

A 50 kg girl standing on a stationary 150 kg boat dives off it horizontally at 3 m/s. Ignoring the water's drag, the boat moves off at:

a 3 m/s the same way

b 3 m/s the opposite way

c 1 m/s the opposite way

d 0, as it is heavier

4.3 Newton's laws of motion

Newton's three laws are the foundation of classical mechanics. The first says what happens when no resultant force acts; the second, how a force changes a body's motion; the third, that forces always come in pairs.

4.3.1 Newton's first law of motion

Statement: every body continues in its state of rest, or of uniform motion in a straight line, unless an external (resultant) force acts on it to change that state.

So a body at rest stays at rest, and a moving body keeps moving at the same speed in the same straight line, unless a resultant force acts. To start a body moving, to stop it, to speed it up or slow it down, or to turn it, needs a force.

- **The first law defines force:** a force is whatever changes, or tends to change, a body's state of rest or of uniform motion in a straight line.
- **The first law is the law of inertia:** it says that bodies by themselves keep their state, which is just what inertia means.

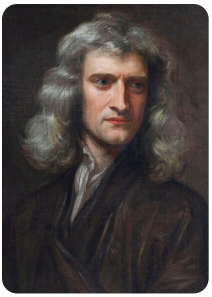


Figure 4.5 *Isaac Newton in 1689, two years after his Principia set out the three laws of motion. Portrait by Godfrey Kneller.*
Photo: Godfrey Kneller, Public domain

Everyday examples:

1. **A bus starts suddenly** and standing passengers fall backwards. The floor pushes their feet forward with the bus, but no forward force acts directly on their upper bodies, which tend to stay at rest.
2. **A bus brakes suddenly** and passengers fall forwards. The floor stops their feet, but their upper bodies tend to keep moving forward.
3. **A bus turns** and the passengers lean to the outside of the turn: their bodies tend to keep going straight.
4. **Mangoes fall** when the branch is shaken: the branch is jerked away but the mango tends to stay at rest, and its stalk breaks.
5. **A long jumper runs up** before jumping: the speed she has at take-off carries her on through the air.
6. **Jumping off a moving bus** (dangerous: never do it). Your feet stop when they touch the road, but your upper body keeps moving forward with the bus's speed, so you fall forward. If you ever must, you would have to land running forward, in the bus's direction.
7. **A fan switched off** slows down and stops; **a ball rolling** on the ground slows down and stops. Neither breaks the first law: **friction** and air resistance are forces, and they are what stop it. On ice the ball rolls much farther; in space, with no friction, it would never stop.

CAREFUL

A moving body doesn't need a force to keep moving

The belief that a moving body needs a force to keep it moving is Aristotle's, and it is wrong. On a road a cyclist must keep pedaling only because friction and air resistance act against her; her pedaling balances them. With the forces balanced, the resultant force is zero and she moves at a steady speed, just as the first law says.

Check 4

A spacecraft far from every star and planet switches off its engines while moving at 10 km/s. It will:

a slow down and stop
b slow down to a steady, smaller speed
c keep moving at 10 km/s in a straight line
d start to fall

4.3.2 Newton's second law of motion

Statement: the rate of change of momentum of a body is directly proportional to the resultant force acting on it, and takes place in the direction of the force.

Deriving F = ma. A body of mass m moving at u is acted on by a constant force F for a time t, and reaches a velocity v.

1. Its momentum changes from mu to mv, so the rate of change of momentum is $\dfrac{mv - mu}{t}$.
2. By the second law, $F \propto \dfrac{mv - mu}{t} = m\,\dfrac{v - u}{t}$.
3. Since $\dfrac{v - u}{t} = a$, the acceleration, $F \propto ma$, that is, $F = kma$, where k is a constant.
4. The unit of force is chosen to make $k = 1$: **one newton is the force that gives a mass of 1 kg an acceleration of 1 m/s²**. Putting $F = 1$ N, $m = 1$ kg and $a = 1$ m/s² gives $1 = k \times 1 \times 1$, so $k = 1$, and

$$\boxed{\vec{F} = m\vec{a}}$$

So **1 N = 1 kg m/s²**. The second law **measures force**: from a body's mass and its acceleration you know the resultant force on it. In calculus form, $\vec{F} = \dfrac{d\vec{p}}{dt}$, which also covers a body whose mass changes (a rocket burning fuel).

The $\vec{F}$ is the **resultant** (vector sum) of all the forces on the body, and the acceleration is in its direction.

WORKED EXAMPLE

Three forces on one body

A 6 kg object is acted on by three forces, $\vec{F}_1 = 20\hat{i} + 30\hat{j}$, $\vec{F}_2 = 8\hat{i} - 50\hat{j}$ and $\vec{F}_3 = 2\hat{i} + 2\hat{j}$ (in newtons). Find its acceleration.

Answer

The resultant force is the vector sum, component by component:

$$\vec{F} = (20 + 8 + 2)\hat{i} + (30 - 50 + 2)\hat{j} = 30\hat{i} - 18\hat{j} \text{ N}$$

$$\vec{a} = \frac{\vec{F}}{m} = \frac{30\hat{i} - 18\hat{j}}{6} = \mathbf{5\hat{i} - 3\hat{j}} \text{ m/s}^2$$

Its size is $\sqrt{5^2 + 3^2} = 5.83$ m/s².

WORKED EXAMPLE

Two blocks pushed together

A 3 kg block touches a 2 kg block on a frictionless floor. A horizontal force of 10 N pushes the 3 kg block, which pushes the 2 kg block. Find the force between the blocks.

Answer

The two move together, so the 10 N accelerates both:

$$a = \frac{F}{m_1 + m_2} = \frac{10}{3 + 2} = 2 \text{ m/s}^2$$

The only horizontal force on the 2 kg block is the push of the first block, which must give it 2 m/s²:

$$P = m_2 a = 2 \times 2 = \mathbf{4} \text{ N}$$

(Check on the 3 kg block: 10 N forward, 4 N back from the second block, resultant 6 N = 3 × 2. ✓)

WORKED EXAMPLE

A car's engine

A car of mass 1000 kg accelerates at 2 m/s². What resultant force acts on it? If the resistance to its motion is 1000 N, what force does the engine provide?

Answer

Resultant force: $F = ma = 1000 \times 2 = \mathbf{2000\ N}$.

The resultant is the engine's force F' minus the resistance: $F' - 1000 = 2000$, so $F' = \mathbf{3000\ N}$. (Strictly, the forward force comes from the road, pushing on the driven tires, Section 4.4.)

WORKED EXAMPLE

A man in a lift

A man of mass 50 kg stands in a lift. Find the force of the floor on him when the lift accelerates at 2 m/s² a. upward, b. downward. (Take $g = 10$ m/s².)

Answer

Two forces act on the man: his weight $mg = 500$ N down, and the floor's push R up. He accelerates with the lift.

a. Accelerating upward, the resultant must be upward: $R - mg = ma$, so $R = m(g + a) = 50 \times 12 = \mathbf{600\ N}$.

b. Accelerating downward: $mg - R = ma$, so $R = m(g - a) = 50 \times 8 = \mathbf{400\ N}$.

It is the floor's push R, not the cable's pull on the lift, that accelerates the man.

Check 5

A 2 kg trolley is pulled with 10 N along a level track against a friction of 4 N. Its acceleration is:

a 5 m/s²
b 3 m/s²
c 7 m/s²
d 2 m/s²

4.3.3 Newton's third law of motion

Think about these:

- **Swimming:** you push the water backward with your hands and feet, and you move forward.
- **Jumping:** you push the ground down and back, and you rise up and forward.
- **Firing a gun:** the bullet goes forward, and the gun kicks back into the shoulder (the **recoil**).
- **Punching a wall:** a concrete wall hurts far more than a cushion, because it pushes back harder on your fist.
- **Walking:** a day's walk on a tarred road tires your feet more than a walk on grass: the hard road pushes back on each footfall over a shorter time, with a bigger force.

In every case there are **two forces, on two different bodies**, equal in size and opposite in direction.

Statement: to every action there is always an equal and opposite reaction. That is, when two bodies interact, **the force that A exerts on B is equal in size and opposite in direction to the force that B exerts on A**:

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

It is also called the **law of action and reaction**. Three facts about the pair matter:

1. **They act on different bodies.** Your hand pushes the wall; the wall pushes your hand. So they **never cancel each other**: forces cancel only when they act on the same body.
2. **They act at the same instant.** Neither comes first; which one we call the "action" is only a name. They start together and stop together.
3. **They are the same kind of force:** both contact forces, or both gravitational, or both magnetic.

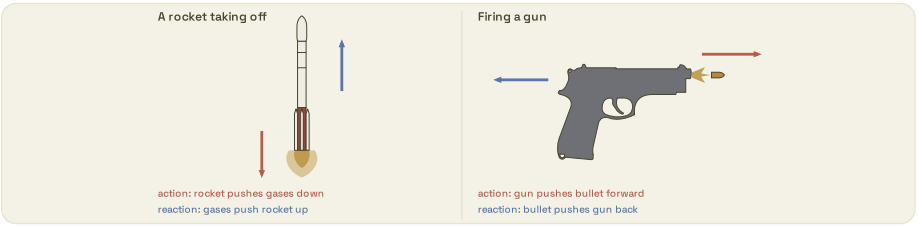


Figure 4.6 *Action and reaction on different bodies. The rocket pushes its exhaust gas down (red), and the gas pushes the rocket up (blue). The gun pushes the bullet forward, and the bullet pushes the gun back: the recoil.*

More examples:

- **Rowing a boat:** the oar pushes the water backward; the water pushes the oar, and the boat, forward.
- **A rocket** pushes its burnt gas backward out of the nozzle at great speed; the gas pushes the rocket forward. It doesn't need air to push against.
- **Walking or a horse pulling a cart:** the foot or hoof pushes back on the ground; the ground pushes forward on the foot (Section 4.4).

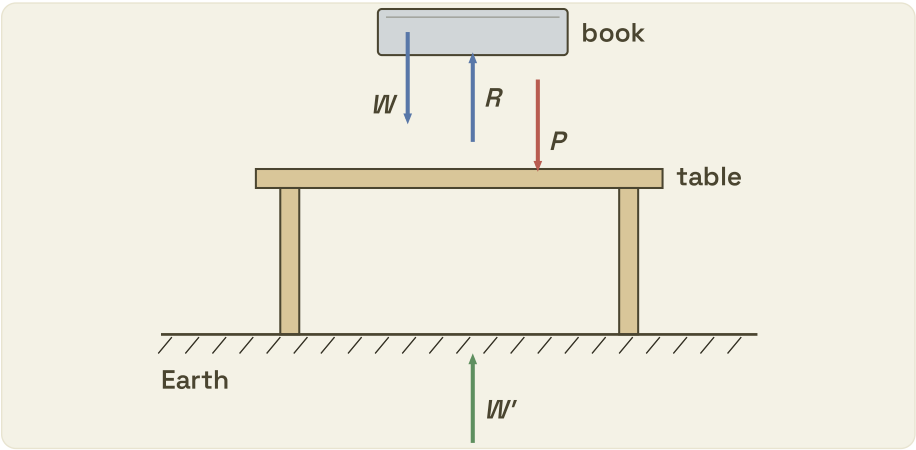


Figure 4.7 *A book at rest on a table, drawn lifted a little off it so the forces can be seen. Two forces act* ***on the book****: its weight W (the Earth pulling it down) and the table's push R (up). They balance, so the book stays at rest. They are* ***not*** *an action–reaction pair. R pairs with the book's push P on the table (down), and W pairs with the book's pull W′ on the Earth (up).*

CAREFUL

Weight and the normal reaction are not an action–reaction pair

Some books say that the weight of a book on a table and the table's upward push on it are "action and reaction". They are not: both act on the **same** body, the book, and they are different kinds of force (one gravitational, one a contact push). They are equal only because the book is in equilibrium; in a lift accelerating upward, the push is bigger than the weight. The reaction to the table's push on the book is the book's push on the table; the reaction to the Earth's pull on the book is the book's pull on the Earth. If an exam asks for "the angle between action and reaction" of a book on a table, the answer is 180°: every action–reaction pair is opposite.

Why the second law is "the real law"

The second law contains the first. Write it as

$$F = ma = \frac{m(v-u)}{t}$$

If no force acts, $F = 0$, so $\dfrac{m(v-u)}{t} = 0$. Since $m \neq 0$, $v - u = 0$, that is $v = u$: the body keeps its velocity, at rest if it was at rest, moving steadily in a straight line if it was moving. That is the first law. If a force acts, $F \neq 0$, so $v \neq u$: the state changes. So the first law follows from the second, and the second is often called the **real law of motion**. (The third law is a separate statement about pairs of forces; together with the second it gives conservation of momentum, Section 4.2.)

Check 6

A heavy truck and a small car collide head-on. During the collision, the force of the truck on the car is:

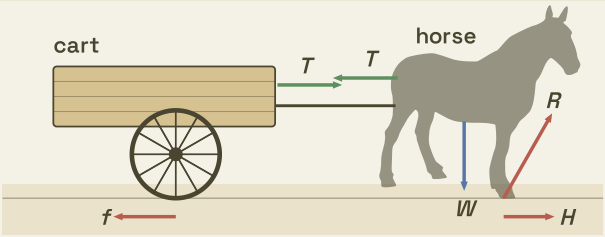
a larger than the force of the car on the truck
b smaller than the force of the car on the truck
c equal in size to the force of the car on the truck
d zero, if the car stops

4.4 Applications of Newton's laws of motion

The method is always the same: pick one body, draw **all** the forces acting **on it** (a **free-body diagram**), find the resultant in the direction of motion, and set it equal to mass × acceleration.

Horse and cart

A horse pulls a cart by pushing its hooves down and back on the ground. By the third law the ground pushes the hooves up and forward with a force R (Figure 4.8). The vertical part of R balances the horse's weight; its forward horizontal part H moves the



Horse: David Orr, CC0, via PhyloPic

Figure 4.8 *A horse pulling a cart. On the horse: its weight W, the ground's push R on its hooves, slanting forward (its parts: an upward part balancing W, and a forward part H), and the rope's pull T back. On the cart: the rope's pull T forward and the ground's friction f back. (The cart's weight and the ground's upward push on its wheel balance each other, and are left out.)*

horse on. The rope pulls the cart forward with a tension T, and pulls the horse back with an equal tension T (third law). Friction f on the cart's wheels acts backward. Let the horse's mass be m, the cart's m', and their common acceleration a.

1. **On the horse**, the horizontal forces are H forward and T back:

$$H - T = ma \tag{4.1}$$

2. **On the cart**, the horizontal forces are T forward and f back (its weight is balanced by the ground's normal reaction R_c):

$$T - f = m'a \tag{4.2}$$

3. **Adding** (4.1) and (4.2), the tensions cancel:

$$H - f = (m + m')a \quad \Rightarrow \quad a = \frac{H - f}{m + m'}$$

Another way: treat horse and cart as **one system**. The rope's tensions are internal, equal and opposite, and cancel. The only external horizontal forces are H forward (the ground on the hooves) and f back (the ground on the wheels), so $H - f = (m + m')a$ at once.

So **the force that moves the horse and cart is provided by the ground**, pushing on the horse's hooves. The system accelerates when $H > f$, moves steadily when $H = f$, and slows down when $H < f$. A horse on ice, whose hooves can't push on the ground, can't pull anything.

Two masses over a pulley

Two masses m_1 and m_2, with $m_1 > m_2$, hang from the ends of a light, inextensible string over a smooth (frictionless) pulley (Figure 4.9). The string has the same tension T all along. The heavier mass accelerates down and the lighter up, with the same acceleration a (the string doesn't stretch).

1. **For** m_1 (accelerating down), the resultant force is down: $W_1 - T = m_1 a$, that is $m_1 g - T = m_1 a$. (4.3)
2. **For** m_2 (accelerating up), the resultant force is up: $T - W_2 = m_2 a$, that is $T - m_2 g = m_2 a$. (4.4)
3. **Adding**, T cancels: $(m_1 - m_2)g = (m_1 + m_2)a$, so

$$a = \frac{(m_1 - m_2)\, g}{m_1 + m_2} = \frac{W_1 - W_2}{m_1 + m_2}$$

4. **Tension:** from (4.4), $T = m_2(g + a) = m_2 g \left(1 + \dfrac{m_1 - m_2}{m_1 + m_2}\right)$, so

$$T = \frac{2 m_1 m_2\, g}{m_1 + m_2}$$

Another way: treat both masses as one system moving along the string. The driving force is the difference of the weights, $W_1 - W_2$, and the mass being moved is $m_1 + m_2$ (the tensions are internal and cancel): $a = \dfrac{W_1 - W_2}{m_1 + m_2}$.

Notice that T lies between the two weights: less than $m_1 g$ (so m_1 falls) and more than $m_2 g$ (so m_2 rises). With equal masses, $a = 0$ and $T = mg$. This arrangement is the **Atwood machine** (George Atwood, 1784): it slows free fall down by the factor $\dfrac{m_1 - m_2}{m_1 + m_2}$, so that g can be measured with a clock.

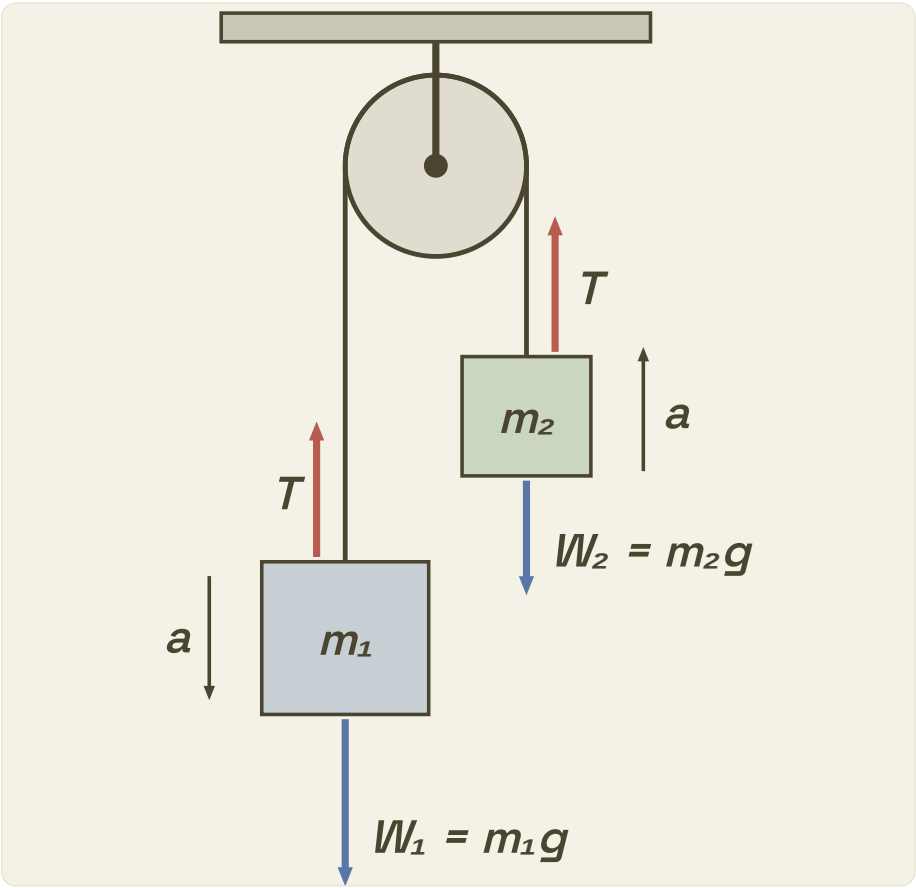


Figure 4.9 *Two masses on a light string over a smooth pulley, with $m_1 > m_2$. The heavier m_1 accelerates down and m_2 up, both at a. The string pulls each up with the same tension T; their weights are $W_1 = m_1 g$ and $W_2 = m_2 g$.*

YOUR TURN

An Atwood machine

Masses of 5 kg and 3 kg hang over a smooth pulley. Find the acceleration and the tension ($g = 9.8$ m/s²).

Answer

$a = \dfrac{(5-3)\times 9.8}{5+3} = \mathbf{2.45\ m/s^2}$; $T = \dfrac{2\times 5\times 3\times 9.8}{8} = \mathbf{36.75\ N}$ (between 3g = 29.4 N and 5g = 49 N, as it must be).

Motion in a lift: apparent weight

When you stand on a weighing scale, it shows how hard it pushes up on you, the normal reaction R, not your mass. In a lift that is accelerating, that push changes, and you feel heavier or lighter: your **apparent weight** is R.

Take a person of mass m standing on the floor of a lift. Two forces act on the person: the weight mg down and the floor's push R up (Figure 4.10).

1. **Lift at rest, or moving up or down at a steady speed** ($a = 0$): $R - mg = 0$, so $R = mg$. The person feels their normal weight.

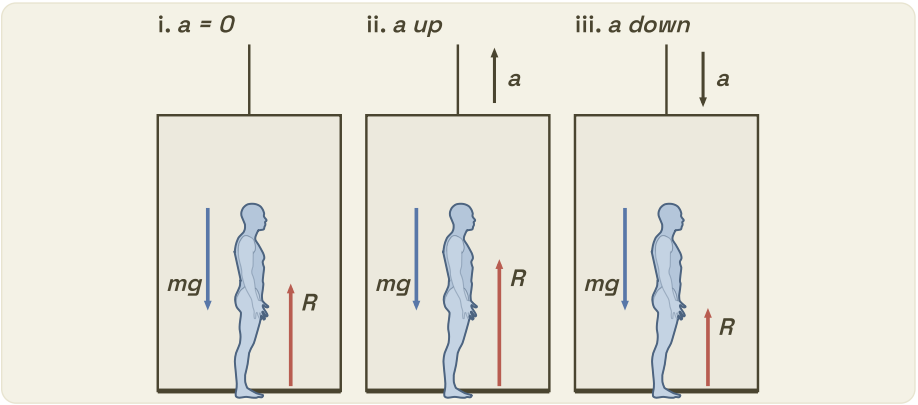


Figure 4.10 *A person of weight mg standing in a lift. The floor's push R is drawn to scale for an acceleration of g/4. i. At rest or moving at a steady speed: R = mg. ii. Accelerating upward: R = m(g + a), larger. iii. Accelerating downward: R = m(g − a), smaller.*

2. **Lift accelerating upward** (speeding up on the way up, or slowing down on the way down): the resultant is upward, $R - mg = ma$, so

$$R = m(g + a) > mg$$

The person feels heavier.

3. **Lift accelerating downward** (speeding up on the way down, or slowing down on the way up): the resultant is downward, $mg - R = ma$, so

$$R = m(g - a) < mg$$

The person feels lighter: an **apparent loss of weight**.

4. **Lift falling freely** (the cable breaks, $a = g$): $R = m(g - g) = 0$. The floor doesn't push at all; the person feels **weightless**, floating with the lift. Astronauts in orbit are weightless in exactly this way: they and their spacecraft are falling together round the Earth.
5. **Lift pulled down with** $a > g$ (faster than free fall): the floor would have to pull the person down, $R = m(g - a) < 0$, but a floor can only push. So the floor drops away from the person's feet, and relative to the lift the person moves up until they reach the **ceiling**, which then pushes them down with a force $m(a - g)$. They are pressed against the ceiling, as if standing on it.

Check 7

Your weighing scale in a lift reads less than your normal weight. The lift could be:

a moving up at a steady speed
b moving down at a steady speed
c moving up and slowing down
d moving down and slowing down

DID YOU KNOW?

Kathmandu's lifts and the first second

The scale's reading changes only while the lift **accelerates**: in the first second or so, as it starts, and in the last second, as it stops. In between, gliding at a steady speed, you feel your normal weight, however fast it goes. Lifts are made to accelerate at about 1 m/s², so a 50 kg person's apparent weight goes from about 490 N to 540 N while starting up: about 10 % more, just enough to feel.

4.5 Moment, torque and equilibrium

The door turns about the line of its hinges, the stone about the edge it tips over. This line is the **axis of rotation**. How strongly a force turns a body about an axis depends on two things: the size of the force, and how far from the axis it acts.

The **moment of a force**, or **torque**, about an axis is the product of the force and the perpendicular distance from the axis to the force's line of action:

$$\tau = rF$$

where r is that perpendicular distance (the **moment arm**). Its SI unit is the newton meter, N m. Torque turns a body about the axis: a larger torque turns it faster and faster (gives it a larger angular acceleration, Grade 12).

In vector form, with $\vec{r}$ the position vector of the point where the force acts, measured from the axis,

$$\vec{\tau} = \vec{r} \times \vec{F}, \qquad \tau = rF\sin\theta$$

where θ is the angle between $\vec{r}$ and $\vec{F}$; $r\sin\theta$ is the perpendicular distance. The torque is greatest when the force is at right angles to $\vec{r}$, and zero when the force acts along $\vec{r}$ (pushing a door straight towards its hinges) or at the axis itself.

So for the same torque, a larger r needs a smaller F: that is why a door's handle is at the edge far from the hinges, why a spanner has a long handle, and why a long bar turns a stone you couldn't move by hand.

Check 8

A spanner 0.25 m long needs a force of 80 N at its end to loosen a nut. With a pipe slipped over it to make it 0.50 m long, the force needed at the end is:

a 160 N
b 40 N
c 80 N
d 20 N

Clockwise and anticlockwise moments

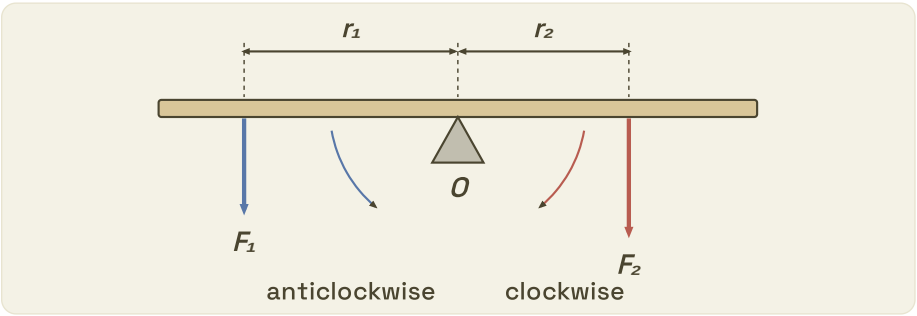


Figure 4.11 *A rod pivoted at its middle, O. F_1 at a distance r_1 to the left tends to turn it anticlockwise; F_2 at r_2 to the right tends to turn it clockwise.*

A rod pivoted at O has two forces on it (Figure 4.11). F_1, on the left, tends to turn it **anticlockwise**; its moment $F_1 \times r_1$ is an **anticlockwise moment**. F_2, on the right, tends to turn it **clockwise**; its moment $F_2 \times r_2$ is a **clockwise moment**. Moments are often given a sign: anticlockwise positive, clockwise negative.

Principle of moments

Let a rod pivoted at O be in equilibrium under several forces. F_1 and F_2 act at A and B and turn it anticlockwise; F_3, F_4 and F_5 act at C, D and E and turn it clockwise (Figure 4.12).

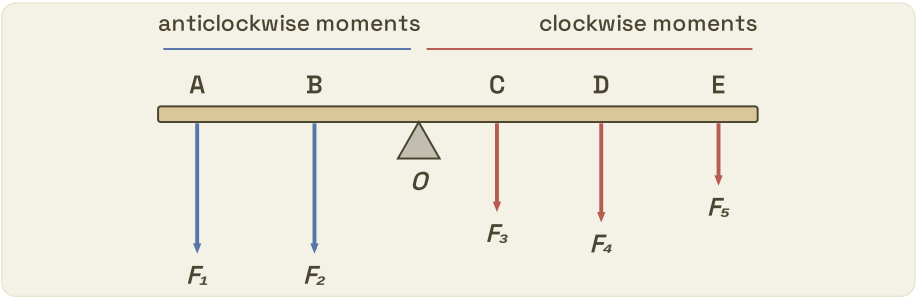


Figure 4.12 *A rod pivoted at O, in equilibrium under five downward forces: F_1 and F_2 at A and B on the left (anticlockwise moments), F_3, F_4 and F_5 at C, D and E on the right (clockwise moments).*

Sum of the anticlockwise moments $= F_1 \times OA + F_2 \times OB$
Sum of the clockwise moments $= F_3 \times OC + F_4 \times OD + F_5 \times OE$

When the rod is in equilibrium, it is found that

$$F_1 \times OA + F_2 \times OB = F_3 \times OC + F_4 \times OD + F_5 \times OE$$

This is the **principle of moments**: **when a body is in equilibrium under several forces, the sum of the clockwise moments about any point equals the sum of the anticlockwise moments about that point**; the resultant moment is zero. A see-saw balances, and a shopkeeper's beam balance weighs, by this principle.

WORKED EXAMPLE

A see-saw

Pratiksha (30 kg) sits 2.0 m from the pivot of a see-saw. Where must her brother Sujan (40 kg) sit to balance her?

Answer

Taking moments about the pivot (the see-saw's own weight acts at the pivot, so it has no moment): $30g \times 2.0 = 40g \times d$, so $d = \dfrac{60}{40} = \mathbf{1.5\ m}$ from the pivot, on the other side. (The g cancels: only the ratio of the weights matters.)

Like and unlike parallel forces

Two parallel forces on a body can be replaced by a single force, their **resultant**, which has the same effect.

- **Like parallel forces** act in the **same** direction. Their resultant is their sum, $R = P + Q$, in the same direction, and it acts at a point C **between** them, nearer the larger force, such that $P \times AC = Q \times BC$.
- **Unlike parallel forces** act in **opposite** directions. If $P > Q$, their resultant is their difference, $R = P - Q$, in the direction of the larger force, and it acts at a point C **outside** them, on the side of the larger force, again with $P \times AC = Q \times BC$.

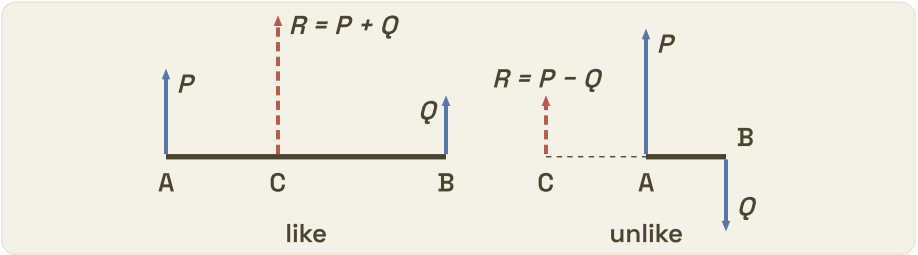


Figure 4.13 Left: like parallel forces P and Q at A and B; their resultant R = P + Q acts at C between them. Right: unlike parallel forces P (up) and Q (down) at A and B; their resultant R = P − Q acts at C, outside AB, beyond the larger force P.

If two unlike parallel forces are **equal**, $P = Q$, their resultant is zero, yet they still turn the body: they form a **couple** (Section 4.5.1).

4.5.1 Torque of a couple

Two equal and opposite parallel forces whose lines of action don't coincide form a **couple**. A couple has no resultant force, so it can't move the body along, but it turns it.

In Figure 4.15, equal forces F act at the ends L and M of a rod pivoted at its midpoint O, one down and one up. Both turn the rod anticlockwise, so the moment of the couple is the sum of the two moments:

$$\tau = F \times OL + F \times OM = F(OL + OM) = F \times LM$$

The moment (torque) of a couple is either force multiplied by the perpendicular distance between their lines of action (the **arm** of the couple). It is the same about every point, not just about O: try it about L, where one force has no moment and the other has $F \times LM$.

Couples are everywhere a thing is turned without being moved along:

- turning a tap, a key in a lock, or a screwdriver: the fingers push one side one way and the other side the other way;
- turning a car's steering wheel with both hands;
- a compass needle in the Earth's magnetic field: the north pole is pulled one way and the south pole the other, so the needle turns until it lies along the field.

WORKED EXAMPLE

A plate held by two couples

A rectangular plate ABCD has AB = 5 m and BC = 3 m. Forces of 100 N act along BC and along DA, in opposite directions. Find the torque on the plate. What forces, acting along CD and AB, will keep the plate in equilibrium?

Answer

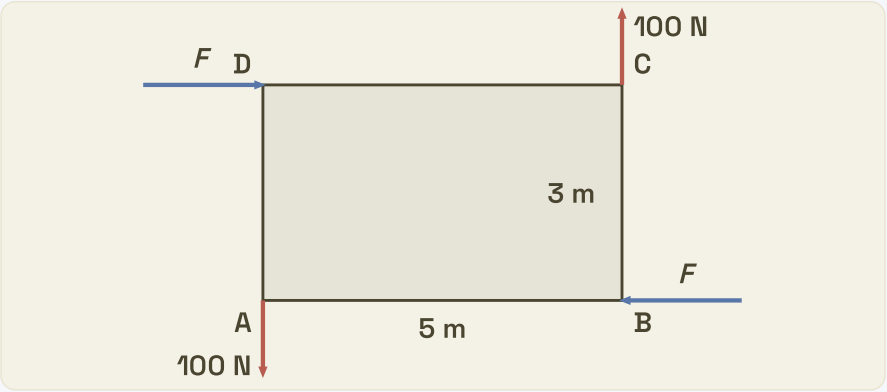


Figure 4.14 *The plate ABCD, AB = 5 m, BC = 3 m. 100 N acts up along BC and 100 N down along DA: a couple of arm AB = 5 m, turning the plate anticlockwise. Equal forces F along DC (pushing D towards C) and along BA (pushing B towards A), 3 m apart, make the balancing clockwise couple.*

The two 100 N forces are equal, parallel and opposite: a couple. The perpendicular distance between their lines of action (BC and DA) is the length **AB = 5 m**, so

$$\tau = F \times d = 100 \times 5 = \mathbf{500}\ \text{N m}$$

To hold the plate in equilibrium we need an equal couple turning the other way. Forces F along CD and AB, in opposite directions, are a couple whose arm is the distance between CD and AB, **BC = 3 m**:

$$F \times 3 = 500 \quad \Rightarrow \quad F = \frac{500}{3} = \mathbf{166.7}\ \text{N}$$

The 100 N forces turn the plate one way; the forces of 166.7 N along CD and AB must turn it the other way. (Each pair adds up to zero force, so the plate is also in translational equilibrium.)

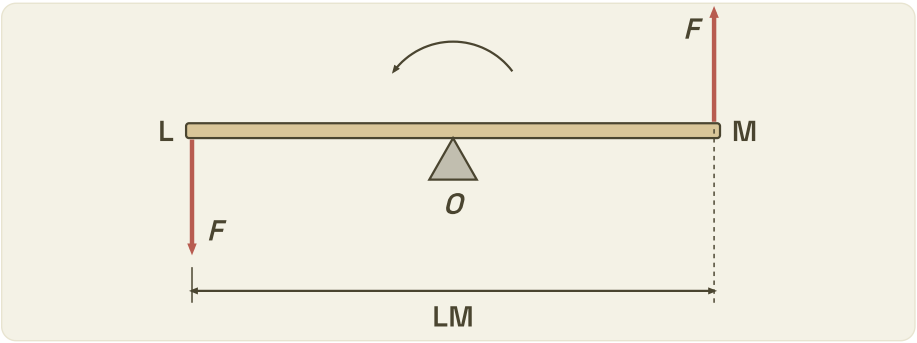


Figure 4.15 *A couple on a rod LM pivoted at its middle, O: F downward at L and F upward at M. Both turn the rod anticlockwise. The perpendicular distance between them is LM.*

Check 9

A driver turns a steering wheel of diameter 0.40 m with two equal and opposite forces of 30 N on opposite sides of the rim. The torque is:

a 6 N m
b 12 N m
c 24 N m
d 0, since the forces cancel

4.5.2 Work done by a couple

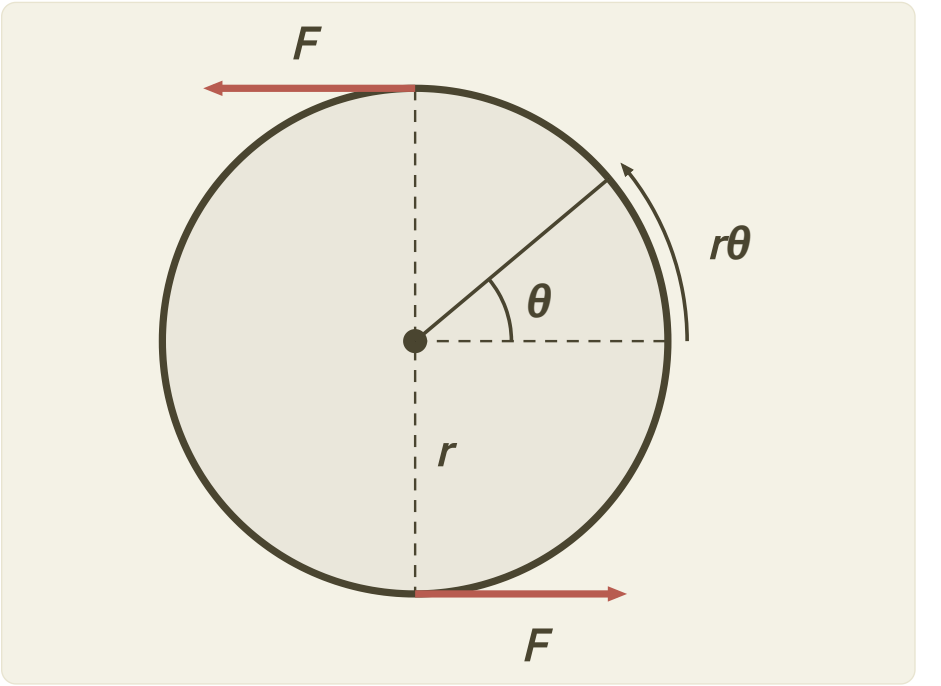


Figure 4.16 *A wheel of radius r turned by a couple: forces F along the rim at the two ends of a diameter. As the wheel turns through an angle θ, every point on its rim, including the two where the forces act, moves an arc rθ.*

Two equal and opposite forces F act along the rim, at the two ends of a diameter of a wheel of radius r (Figure 4.16), and the wheel turns through an angle θ (in radians).

1. Each force stays along the rim, so it moves its point of application along an arc of length $r\theta$ in its own direction. The work done by each force is $F \times r\theta$.
2. The total work done by both forces is $2Fr\theta = F \times 2r \times \theta$.
3. $F \times 2r$ is the torque of the couple (its arm is the diameter $2r$):

$$\boxed{W = \tau\theta}$$

The work done by a couple is its torque multiplied by the angle turned through (in radians). This is the turning version of "work = force × distance" (Unit 5).

4.5.3 Equilibrium

A body is in **equilibrium** when its state of motion doesn't change: the forces on it produce no acceleration and no turning.

- **Translational equilibrium:** the resultant (vector sum) of all the forces on the body is zero, $\sum \vec{F} = 0$. The body then has no acceleration: it is **at rest, or moving with a constant velocity**. The forces balance each other.
- **Rotational equilibrium:** the sum of the torques on it about any point is zero, $\sum \tau = 0$: the clockwise and anticlockwise torques balance (the principle of moments). The body doesn't start to turn, or keeps turning at a steady rate.

Concurrent forces are forces whose lines of action all pass through one point. They have no moment about that point, so a body under concurrent forces is in equilibrium when their resultant is zero (three such forces then form a closed triangle, Unit 2).

Equilibrium of rigid bodies

A **rigid body** is one in which the distance between any two of its points stays the same, however the body moves: it doesn't stretch, bend or squash. (No real body is perfectly rigid, but a steel beam or a cricket bat is close enough for most purposes.) Unlike a point, a rigid body can turn as well as move along, so it needs **both** conditions:

1. **The vector sum of all the forces on it is zero**, $\sum \vec{F} = 0$. Then $m\vec{a} = 0$; since $m \neq 0$, $\vec{a} = \dfrac{d\vec{v}}{dt} = 0$, so $\vec{v}$ is constant: the body is at rest or moves with constant velocity (**translational equilibrium**).
2. **The sum of the torques on it about any point is zero**, $\sum \tau = 0$. In Grade 12 you will meet $\tau = I\alpha$ (torque = moment of inertia × angular acceleration, the turning version of $F = ma$); with $\sum \tau = 0$, $\alpha = \dfrac{d\omega}{dt} = 0$, so the angular velocity ω is constant: the body is not turning, or turns at a steady rate (**rotational equilibrium**).

A ladder leaning against a wall, a bridge, a loaded beam and the mast of Question 42 are all solved with these two conditions: the forces add up to zero, and the moments about a well-chosen point add up to zero.

CAREFUL

Equilibrium isn't only rest

Some books define equilibrium as "the body is at rest". A body in equilibrium can also be **moving at a constant velocity**: a car cruising at a steady 60 km/h on a straight level road is in equilibrium, its engine's forward force exactly balancing friction and air resistance.

Check 10

A gymnast holds a still handstand. Which is true?

a Only the forces on her must balance
b Only the moments about her hands must balance
c Both the forces and the moments must balance
d Neither, since she is not moving

4.6 Solid friction

When two solid surfaces touch and one slides, or tries to slide, over the other, a force appears that opposes their relative motion. This resistance is **friction**, and the force is the **frictional force**. It acts along the surfaces in contact.

4.6.1 Laws of solid friction and their verification

Introduction. Look at these everyday things:

1. Walking on a tarred road is easy; on a muddy path, or on ice, your feet slip.
2. A pen writes easily on paper, but slides about and hardly writes on a sheet of plastic.
3. A nail driven into a wall stays put and can hold a load.
4. A fan switched off slows down and stops.
5. A ball rolling on the ground slows down and stops.

In each there is something that opposes motion between surfaces: friction. Friction is what gives your shoe grip, makes a nail hold, and stops the fan and the ball (which is why they don't break the first law: friction is the force that changes their motion).

Cause of friction. Friction comes from the interactions of the atoms and molecules where the two surfaces touch. It acts along the surface, against the relative motion. It was explained in two ways:

1. **The old view: interlocking.** Every surface, however smooth it looks, is rough at a small scale (Figure 4.17). The bumps of one surface catch in the hollows of the other, and to slide, one must be lifted over or the bumps broken off. This is why sandpaper and a rough road give more friction.
2. **The modern view: adhesion.** The bumps touch only at their tips, so the true area of contact is very small, and the pressure there is huge. At these points the molecules of the two surfaces come close enough to **attract one another** and stick (adhere): tiny "cold welds". To slide, these welds must be broken, again and again. A heavier load presses the surfaces closer, flattens the tips and increases the true area of contact, which is why friction grows with the load (the sixth law below). And it explains why two very smooth, clean surfaces (polished glass on glass, clean metal on metal in a vacuum) can have **more** friction, not less: more of them touches.

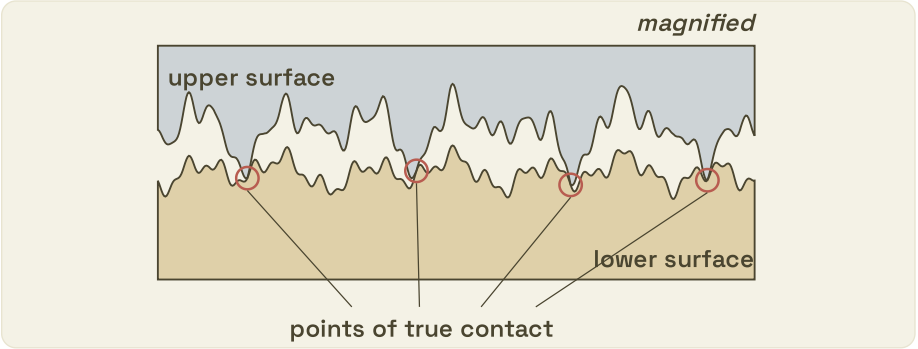


Figure 4.17 *Two surfaces seen under a microscope. Even polished surfaces are rough at this scale: they touch only at the tips of a few bumps, and the true area of contact is a tiny fraction of the area that seems to touch. At those tips the atoms of the two surfaces attract and stick.*

DID YOU KNOW?

Friction welding

Spin one steel rod against another, press them together, and friction heats the contact so much that the metals soften and fuse: **friction welding** joins parts of car engines and aircraft. In space, two clean metal parts that touch can stick together on their own ("cold welding") because there is no layer of air or oxide between them, a problem engineers designing satellites have to plan for.

4.7 Types of friction

Static and kinetic friction

- **Static friction** is the friction between two surfaces in contact that are **not** sliding over each other. It **adjusts itself**: it is exactly as large as it needs to be to stop sliding, and no larger. Push the box with 50 N and static friction is 50 N; push with 100 N and it is 100 N, so the box stays still.
- **Limiting friction** is the **greatest** value static friction can reach, at the moment the body is just about to slide.
- **Kinetic (dynamic) friction** is the friction between two surfaces that **are** sliding over each other. It is roughly constant, and it is **smaller than limiting friction**.

The graph in the widget shows the whole story (Figure 4.18): as the pull rises from zero, the friction rises with it along a 45° line (static friction equals the pull); at the limiting friction the block starts to slide, and the friction drops to the kinetic value and stays there.

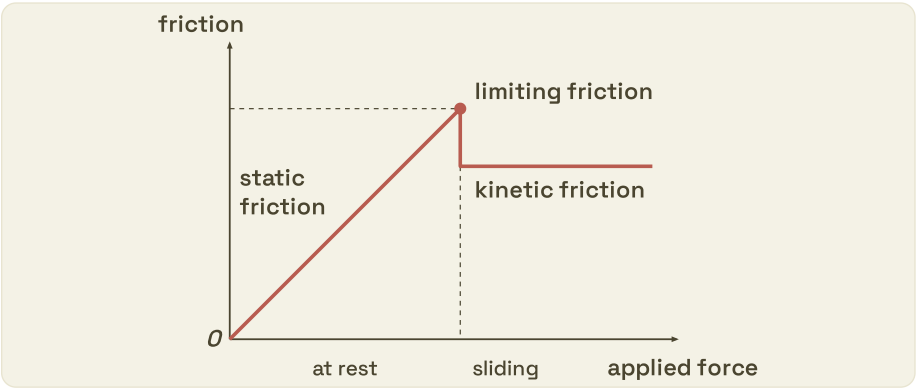


Figure 4.18 *Friction against the applied force for a block on a table. Static friction equals the pull, up to the limiting friction. Then the block slides, and the friction drops to the smaller, steady kinetic friction.*

Sliding and rolling friction

- **Sliding friction** is the kinetic friction when one body **slides** over another: a book pushed across a table, a sledge on snow.
- **Rolling friction** is the resistance when one body **rolls** over another: a wheel on a road, a ball on the ground. It comes mainly from the slight squashing of the wheel and the surface where they touch, which never quite springs back with all its energy.

Rolling friction is far smaller than sliding friction, often tens or even hundreds of times smaller. That is why the wheel was such an invention, why heavy machines are moved on rollers, and why **ball bearings** are fitted in bicycle hubs, fans and motors: they turn sliding into rolling.



Figure 4.19 *A ball bearing. The wheel's axle turns in the inner ring; the balls roll between the rings instead of the axle sliding in its hole.*
Photo: Solaris2006, CC BY-SA 3.0

Check 11

A heavy cupboard is easier to keep moving than to start moving because:

a its mass falls once it moves
b kinetic friction is less than limiting friction
c static friction disappears once it moves
d the floor's normal reaction falls once it moves

4.8 Laws of friction

1. Friction between two surfaces **opposes their relative motion** (or the motion they tend to have).
2. Friction depends on the **nature and roughness** of the two surfaces.
3. The frictional force acts **parallel to the surfaces** in contact (along them).
4. Limiting friction is **independent of the area** of the surfaces in contact (for the same load).
5. Kinetic friction is (nearly) **independent of the relative speed** of the surfaces.
6. Limiting friction is **directly proportional to the normal reaction** between the surfaces: $F \propto R$.

From the sixth law,

$$F = \mu R$$

where the constant μ is the **coefficient of friction** of that pair of surfaces, $\mu = F/R$. It has no unit. For limiting friction, $\mu_s = F_s/R$ (the coefficient of static friction); for kinetic friction, $\mu_k = F_k/R$ (the coefficient of kinetic friction). Since kinetic friction is smaller than limiting friction, $\mu_s > \mu_k$.

Surfaces	μ_s (about)	μ_k (about)
Rubber tire on dry road	1.0	0.8
Rubber tire on wet road	0.7	0.5
Wood on wood	0.5	0.3
Steel on steel (dry)	0.7	0.6
Steel on ice	0.03	0.02
Joint in the human body	0.01	0.003

CAREFUL

F = μR is the largest friction, not always the friction

Some books write F = μR for static friction in general. Static friction adjusts itself from zero up to a largest value, so $F_s \leq \mu_s R$: the equality holds only at the moment of slipping (limiting friction). A 10 kg box on a floor with $\mu_s = 0.5$, pushed gently with 20 N, has a friction of 20 N on it, not $0.5 \times 98 = 49$ N. Likewise, it is **limiting** friction, not static friction in general, that is greater than kinetic friction.

Friction is a necessary evil.

- *Necessary:* without friction we couldn't walk (our feet would slip), hold a pen or write on paper, drive a nail into a wall, tie a knot, or start, steer or stop a vehicle. Brakes, belts and shoes all work by friction.
- *Evil:* friction turns useful energy into heat, wastes fuel, and wears out tires, shoe soles, gears and engines. Because of it, no machine is 100 % efficient.

So we increase friction where we need it (tread on tires, sand on icy roads, grips on shoes) and reduce it where we don't (oil and grease, ball bearings, smooth polished surfaces, streamlined shapes).

Verifying the laws

a. Limiting friction is proportional to the normal reaction (F ∝ R). A wooden block A rests on a horizontal table (Figure 4.20). A light string tied to it runs horizontally over a smooth pulley at the table's edge to a scale pan. Add small weights to the pan until the block just begins to slide: the weight of the pan and its load is then the limiting friction. Now put a second, identical block B on top of A, doubling the normal reaction, and repeat: the load needed to start them sliding doubles. With three blocks it triples. So limiting friction ∝ normal reaction.

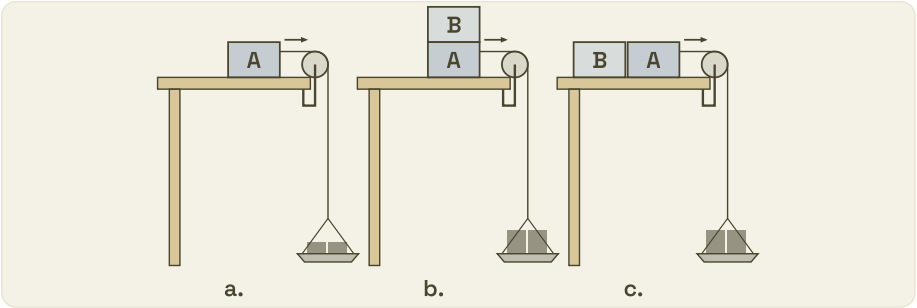


Figure 4.20 *Verifying the laws of friction. a. Block A on a level table is pulled by a string over a pulley to a scale pan; weights are added until it just slides. b. Block B on top of A: the normal reaction doubles, and so does the load needed. c. The same two blocks side by side: twice the area of contact, but the same load is needed as in b.*

b. Limiting friction is independent of the area of contact. Now place the two blocks **one after the other**, side by side on the table (c), instead of stacked (b). The normal reaction is the same, but the area of contact has doubled. The load needed to start them sliding is the same as in (b). So limiting friction doesn't depend on the area of contact.

4.9 Angle of friction

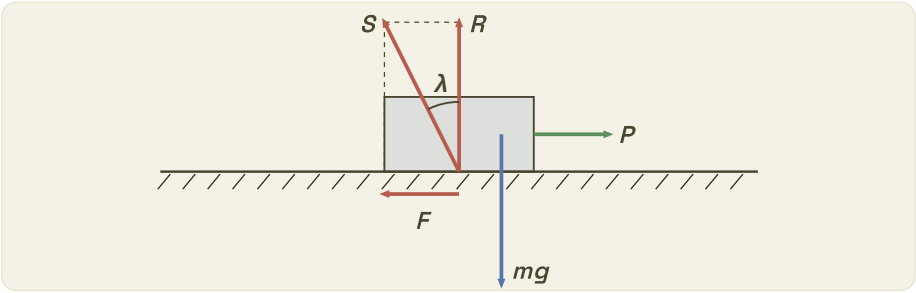


Figure 4.21 *A block on a horizontal surface, pulled by a force P and just about to slide. The normal reaction R (up) and the limiting friction F (back) combine into a single resultant S. The angle λ between S and R is the angle of friction.*

A block of mass m rests on a horizontal surface, and is pulled sideways until it is just about to slide (Figure 4.21). The surface exerts two forces on it: the normal reaction R ($= mg$ here) and the limiting friction F. Their resultant, S, leans back from the normal.

The **angle of friction**, λ, is the angle between the resultant of the limiting friction and the normal reaction, and the normal reaction itself. From the right-angled triangle of forces,

$$\tan\lambda = \frac{F}{R} = \mu$$

The tangent of the angle of friction equals the coefficient of (limiting) friction.

4.10 Angle of repose

The **angle of repose**, θ, is the angle of an inclined plane at which a body placed on it **just begins to slide down**.

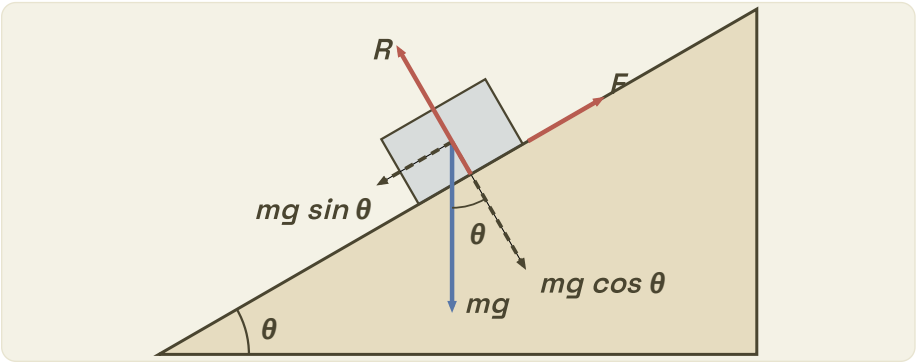


Figure 4.22 *A block on a plane tilted at θ, the angle at which it just begins to slide. Its weight mg splits into mg sin θ down the slope and mg cos θ into the slope. The normal reaction R balances mg cos θ; the limiting friction F, up the slope, balances mg sin θ.*

A block of mass m lies on a plane inclined at θ (Figure 4.22). Its weight mg acts vertically down; split it into two parts:

- $mg\sin\theta$, **down** the slope, which tends to make it slide;
- $mg\cos\theta$, **into** the slope, at right angles to it.

1. Into the slope, the block doesn't move, so the normal reaction balances that part: $R = mg\cos\theta$.
2. Along the slope, when the block is just about to slide (at the angle of repose), the limiting friction F, up the slope, balances the pull down the slope: $F = mg\sin\theta$.
3. Dividing, $\dfrac{F}{R} = \dfrac{mg\sin\theta}{mg\cos\theta} = \tan\theta$. And at the limit $F/R = \mu$, so

$$\tan\theta = \mu$$

The tangent of the angle of repose equals the coefficient of friction. Since also $\tan\lambda = \mu$, **the angle of friction equals the angle of repose**: $\lambda = \theta$.

The angle of repose doesn't depend on the mass of the block, since m cancels. It is why every heap of sand, gravel or grain has the same slope whatever its size (dry sand about 34°), and it matters in Nepal's hills: a slope cut steeper than the angle of repose of its soil, and wetted by the monsoon (which lowers μ), slides. That is one reason landslides follow new roads cut into the hillsides.

YOUR TURN

Find μ from a slope

A coin on a book cover starts to slide when the cover is tilted to 20°. Find the coefficient of static friction.

Answer

$\mu_s = \tan 20^\circ = \mathbf{0.36}$.

Check 12

A block rests on a plank with $\mu_s = 0.5$. The plank is slowly tilted. The block starts to slide at about:

a 30°
b 27°
c 45°
d 60°

IN SHORT

Key ideas of Unit 4

- **Inertia** is a body's resistance to a change in its motion; **mass** measures it. Inertia of rest, of motion, of direction.
- **Momentum** $\vec{p} = m\vec{v}$ (kg m/s). **Impulse** $= \vec{F}\Delta t = \Delta\vec{p}$, the area under a force–time graph; a longer stopping time means a smaller force.
- **Conservation of momentum:** with no external force, $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ (from the second and third laws).
- **First law:** no resultant force, no change in velocity. **Second law:** $\vec{F} = \dfrac{d\vec{p}}{dt} = m\vec{a}$, 1 N = 1 kg m/s². **Third law:** $\vec{F}_{AB} = -\vec{F}_{BA}$, on different bodies, at the same instant.
- Horse and cart: $a = \dfrac{H - f}{m + m'}$; pulley: $a = \dfrac{(m_1 - m_2)g}{m_1 + m_2}$, $T = \dfrac{2m_1m_2g}{m_1 + m_2}$; lift: $R = m(g \pm a)$, zero in free fall.
- **Torque** $\tau = rF = |\vec{r} \times \vec{F}|$ (N m); principle of moments; a **couple**'s torque is F × arm; its work $W = \tau\theta$.
- **Equilibrium:** $\sum \vec{F} = 0$ and $\sum \tau = 0$: at rest or at constant velocity.
- **Friction:** static (self-adjusting, up to limiting), limiting $F = \mu_s R$, kinetic $F = \mu_k R$, $\mu_s > \mu_k$; independent of area. Angle of friction λ and angle of repose θ: $\tan\lambda = \tan\theta = \mu$.

Exercise

Choose the correct option

Check 13

1. What makes mangoes fall when the branch of a mango tree is shaken?

a inertia of rest
b inertia of motion
c inertia of direction
d the effect of gravity alone

Check 14

2. A body is moving at a constant speed along a straight path. In which case is no force required?

a to increase its speed
b to change its direction
c to keep it moving with uniform velocity
d to decrease its momentum

Check 15

3. When two surfaces are coated with lubricating oil, they:

a slide over each other easily
b stick to each other
c roll over each other
d repel each other

Check 16

4. A force $\vec{F} = 6\hat{i} + 8\hat{j}$ N gives a body an acceleration of 1 m/s². The body's mass is:

a 8 kg
b 9 kg
c 10 kg
d 12 kg

Check 17

5. A person jumps off a moving bus. In which direction does he tend to fall?

a in the direction the bus is moving
b opposite to the direction of the bus
c forward, at right angles to the bus's direction
d straight down, where his feet touch

Check 18

6. A 6 kg object is acted on by the three forces $\vec{F}_1 = 20\hat{i} + 30\hat{j}$, $\vec{F}_2 = 8\hat{i} - 50\hat{j}$ and $\vec{F}_3 = 2\hat{i} + 2\hat{j}$ (in newtons). Its acceleration is:

a $3\hat{i} - 5\hat{j}$
b $5\hat{i} - 3\hat{j}$
c $5\hat{i} + 3\hat{j}$
d $3\hat{i} + 5\hat{j}$

Check 19

7. A 50 kg man pushes on a 40 kg man with a force of 40 N. What force does the 40 kg man exert on the 50 kg man?

a 40 N
b 60 N
c 50 N
d 100 N

Check 20

8. A force of 6 N acts on a 1 kg body at rest, until its velocity is 30 m/s. For how long did the force act?

a 10 s
b 5 s
c 8 s
d 6 s

Check 21

9. Which of these is not a force?

a tension
b impulse
c thrust
d weight

Answer these questions

21. a. State Newton's first law of motion.

Check 22

10. Friction is:

a. a conservative force
b. a non-conservative force
c. a path-independent force
d. an unnecessary force

Check 23

11. A lift moves up with an acceleration equal to $\frac{1}{5}$ of g. The apparent weight of a 60 kg man standing in it is ($g = 10\ m/s^2$):

a. 720 N
b. 480 N
c. 600 N
d. 120 N

Check 24

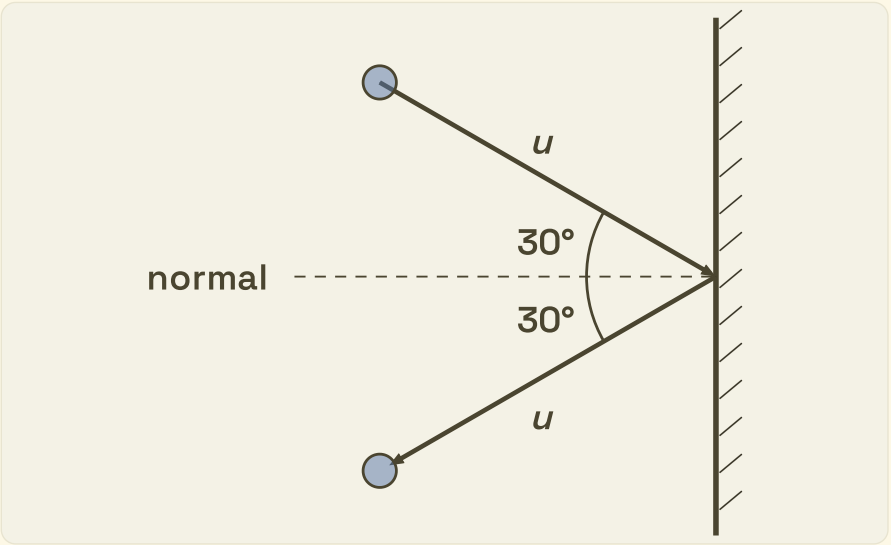


Figure 4.23 *For question 12: a ball meets a wall at 30° to the normal and leaves at 30° on the other side, at the same speed u.*

12. A ball of mass m meets a wall at speed u at 30° to the normal, and bounces off at the same speed, at 30° on the other side of the normal (the picture above). The time of contact is t. The average force on the wall is:

a. $mu\cos 30^\circ/t$
b. $mu\sin 30^\circ/t$
c. $2mu\cos 30^\circ/t$
d. $2mu\sin 30^\circ/t$

Check 25

13. A book lies on a table. The angle between an action and its reaction is:

a. 180°
b. 90°
c. 0°
d. 45°

Check 26

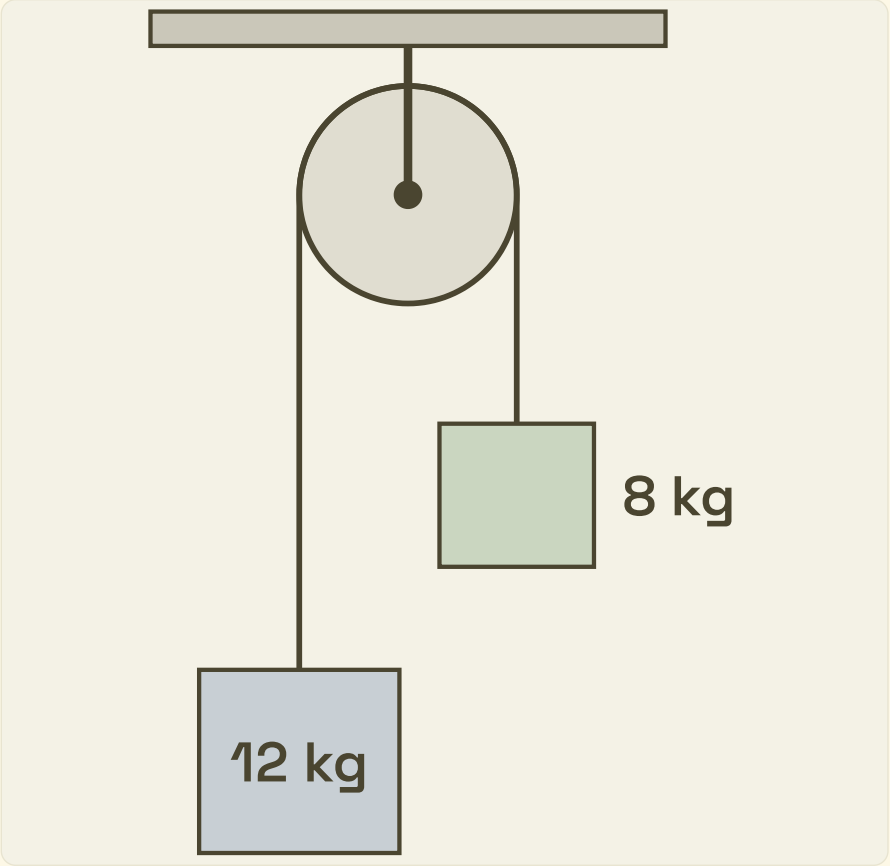


Figure 4.24 *For question 14: blocks of 8 kg and 12 kg hang from the two ends of a light string over a frictionless pulley.*

14. Blocks of 8 kg and 12 kg hang from the ends of a light, inextensible string over a frictionless pulley (the picture above). The acceleration of the system is ($g = 10\ m/s^2$):

a $2\ m/s^2$ b $2.5\ m/s^2$
c $1.5\ m/s^2$ d $1.75\ m/s^2$

Check 27

15. The coefficient of friction between a block and an inclined plane is μ. The angle of repose θ is:

a $\theta = \tan^{-1}\mu$ b $\theta = \cot^{-1}\mu$
c $\theta = \sin^{-1}\mu$ d $\theta = \cos^{-1}\mu$

Check 28

16. A horizontal force of 100 N pulls a 10 kg block along a rough horizontal surface whose coefficient of friction is 0.5. The acceleration is ($g = 10\ m/s^2$):

a $10\ m/s^2$ b $5\ m/s^2$
c $15\ m/s^2$ d $0.5\ m/s^2$

Check 29

18. A 2 kg block rests on a plane inclined at 30°; the coefficient of friction is 0.7. The limiting friction on it is ($g = 9.8\ m/s^2$):

a 10.2 N
b 11.2 N
c 11.9 N
d 9.8 N

Check 30

19. A block slides down a smooth plane inclined at 30°. Its acceleration is ($g = 10\ m/s^2$):

a 2.5 m/s^2
b 10 m/s^2
c 5 m/s^2
d 9.8 m/s^2

Check 31

20. Which best explains why an empty bucket is blown over in a storm?

a its low inertia
b its high inertia
c low friction
d high air resistance

27.

a. State the principle of conservation of linear momentum and explain it from Newton's third law.
b. A superhero throws a boulder of weight 2400 N. What horizontal force must he apply to give it a horizontal acceleration of 12.0 m/s^2?
c. If a friction of 60 N opposes the motion, what horizontal force gives the same acceleration?

28.

a. Define impulse.
b. The graph shows the force on an object against time. Find the impulse between 0 and 10 s.
c. If the object's mass is 2 kg, find its acceleration at i. 2 s and ii. 6 s.

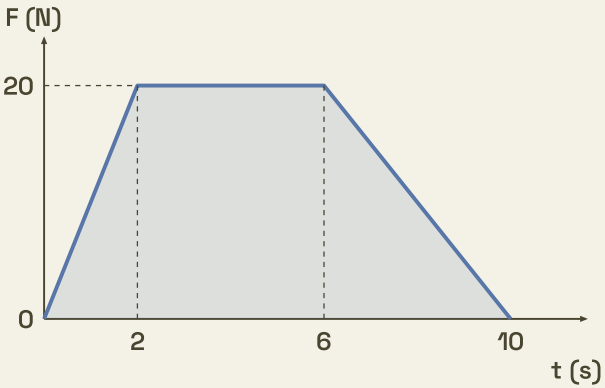


Figure 4.25 *For question 28: the force rises steadily from 0 to 20 N in the first 2 s, stays at 20 N until 6 s, then falls steadily to 0 at 10 s.*

29. a. Define the angle of friction and the angle of repose, and show that they are equal.
    b. A 1 kg block on a plane inclined at 30° starts to slide down from rest. If the coefficient of kinetic friction is 0.2, find its velocity after 5 s.
    c. Find its velocity one second later, at 6 s.

Numerical problems

30. A little wagon of mass 7 kg moves in a straight line on a frictionless horizontal surface at 4 m/s. It is then pushed 3 m in the direction of its velocity by a force of 10 N. Find the acceleration produced and the wagon's final speed.
31. Find the acceleration of a block sliding down a plane inclined at 45° to the horizontal, if the coefficient of sliding friction is 0.3 ($g = 9.8\ m/s^2$).
32. A 150 g wooden block rests on an inclined plane; the coefficient of static friction is 0.30. Find a. the greatest angle to which the plane can be tilted without the block slipping, and b. the force parallel to the plane needed to stop it slipping when the plane is at 30° ($g = 9.8\ m/s^2$).
33. A shot-putter moves a 7 kg shot from rest through 2.8 m and releases it at 13 m/s. Find the average force on the shot.
34. A 5 kg block on a horizontal table is tied by a cord, over a light frictionless pulley at the table's edge, to a hanging 5 kg block. The coefficient of kinetic friction between the block and the table is 0.5. Find the acceleration and the tension in the cord ($g = 9.8\ m/s^2$).

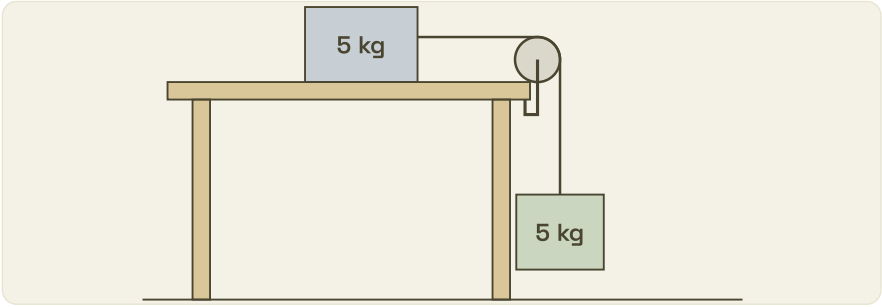


Figure 4.26 *For question 34: a 5 kg block on a table, tied over a pulley at the edge to a hanging 5 kg block.*

35. A shot of mass 7.30 kg is released at 15.0 m/s at 40° above the horizontal. Find the horizontal and vertical components of its momentum.
36. Two skaters collide and hold on to each other on frictionless ice. One, of mass 70.0 kg, moves to the right at 2.00 m/s; the other, of mass 65.0 kg, moves to the left at 2.50 m/s. Find their velocity just after the collision.
37. A bullet of mass 1.80 g moving at 350 m/s strikes a tree and penetrates 0.130 m. Assuming a constant retarding force, find a. the time it takes to stop and b. the force the tree exerts on it.
38. Two dogs pull horizontally on ropes tied to a post, with 60.0° between the ropes. Dog A pulls with 270 N and dog B with 300 N. Find the size of the resultant force and the angle it makes with dog A's rope.
39. A hockey puck of mass 0.160 kg is at rest at the origin on frictionless ice. From $t = 0$ a player pushes it with 0.250 N along the x-axis until $t = 2.00$ s. a. Find its position and speed at 2.00 s. b. The same force is applied again from $t = 5.00$ s. Find its position and speed at 7.00 s.
40. A 12.0 kg chair stands on a rough horizontal floor. You push it with a force of 40.0 N directed at 37.0° below the horizontal, and it slides. a. Draw a labeled free-body diagram for the chair. b. Find the normal force of the floor on the chair.
41. A baseball of mass 0.145 kg moves in the +y direction at 1.30 m/s, and a tennis ball of mass 0.0570 kg moves in the −y direction at 7.80 m/s. Find the size and direction of the total momentum of the two balls.
42. A radio aerial on top of a 20 m mast pulls horizontally on it with 600 N. The mast is held by a stay wire from a point 6 m below the top to the ground, at 60° to the horizontal. Taking the ground's action on the mast as a single force at its base, find the tension in the stay wire.

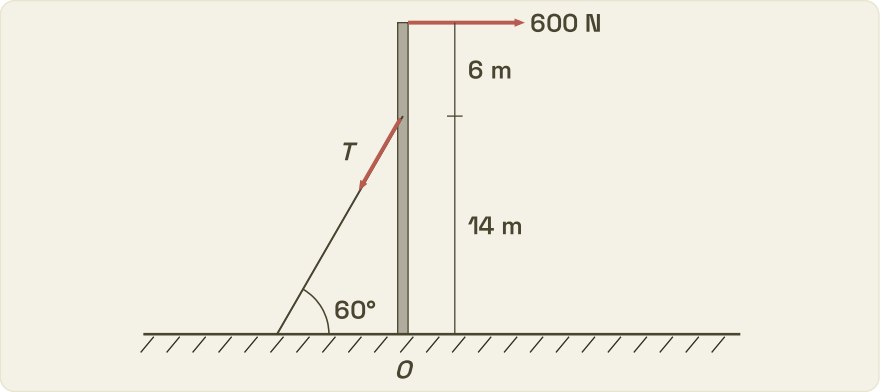


Figure 4.28 *For question 42: the 20 m mast, the aerial's 600 N pull at the top, and the stay wire from 14 m up to the ground at 60° to the horizontal.*

43. A typical car weighs about 12 000 N (a mass of about 1200 kg). If the coefficient of rolling friction is 0.015, find the horizontal force needed to keep it moving at a constant 72 km/h on a level road, and the power the engine must develop to keep up this speed (ignoring air resistance).

44. A roller of diameter 1.0 m weighs 360 N. Find the horizontal force, applied at its center, needed to pull it over a brick 0.1 m high.

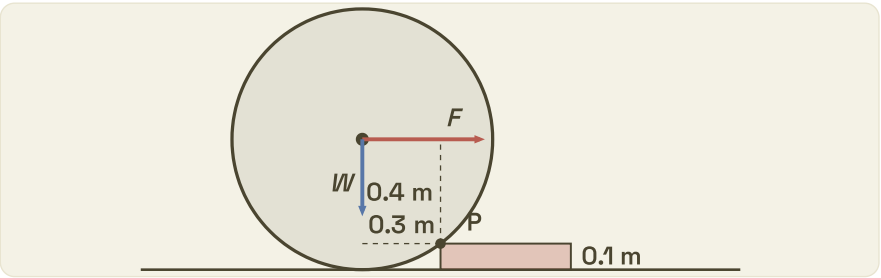


Figure 4.29 *For question 44: the roller just lifting off the ground at the brick's corner P, with its weight W, the pull F at its center, and the arms of both about P.*

45. A block of mass m_1 on a smooth plane inclined at θ is tied by a light string, over a pulley at the top of the plane, to a hanging block of mass m_2.

    a. Find the acceleration of the system in terms of m_1, m_2, θ and g.

    b. Find the tension in the string.

    c. If $m_1 = m_2$ and $\theta = 25°$, find the acceleration.

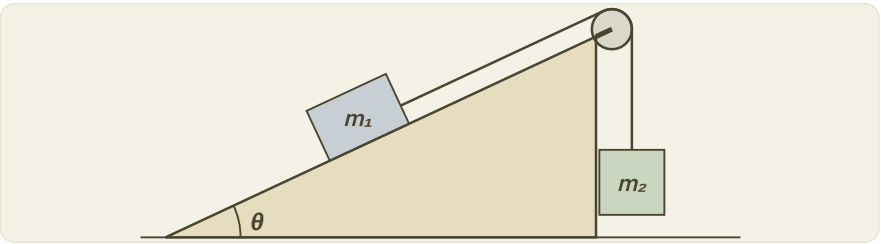


Figure 4.30 *For question 45: block m_1 on a smooth slope at θ, tied over a pulley at the top to the hanging block m_2.*

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.