32. A projectile is fired at 320 m/s at 30° to the horizontal. Find i. the time to reach the greatest height, ii. its horizontal range, iii. the greatest range possible at the same speed ($g = 10\ m/s^2$).

33. A ball is dropped from a height of 60 m. At the same instant, an identical ball is thrown straight up from the ground at 20 m/s along the same vertical line. When and where do they meet? ($g = 10\ m/s^2$)

34. A body moving with uniform acceleration has a velocity of 25 m/s at $t = 5$ s and 34 m/s at $t = 8$ s. How far does it travel in the 12th second?
The textbook gives a body that begins its motion from rest, but a body starting from rest with these two velocities is impossible: from rest, 25 m/s after 5 s means 5 m/s², and then it would have 40 m/s at 8 s, not 34 m/s. The book's numbers don't work with "from rest"; without it they do, and the body was already moving at 10 m/s at t = 0.

35. An object is thrown at 10 m/s at an angle of $\pi/4$ (45°) to the horizontal. Find its range ($g = 10\ m/s^2$). If it is thrown the same way on the Moon, where the acceleration due to gravity is g/6, how does the range change?

36. In a game of cricket, a batter hits the ball at 30 m/s at 30° to the horizontal. The boundary is 75 m away. Will the ball go for a six? (Ignore air resistance; $g = 10\ m/s^2$.)

37. A long jumper leaves the ground at 27° and lands 7.80 m away ($g = 9.8\ m/s^2$).
    a. What was her take-off speed?
    b. If her speed were 5% greater, how long would the jump be?

38. A long jumper leaves the ground at 20° above the horizontal at 11 m/s. How far does he jump, and what is the greatest height he reaches? ($g = 9.8\ m/s^2$)

39. A footballer kicks a ball from the ground at 27 m/s at 30° above the horizontal. Ignoring air resistance, find a. the ball's time in the air (its hang time), b. its greatest height, and c. how far it goes before hitting the ground. ($g = 9.8\ m/s^2$)

40. A rock is thrown from the top of a 50.0 m high cliff at 7.0 m/s, 53.0° above the horizontal. Find its velocity when it hits the ground below. ($g = 9.8\ m/s^2$)

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

Unit 3

Kinematics

This unit online, with its interactives:
science-playground.com/s/11f3

A batter at Kirtipur's cricket ground lofts the ball at 30 m/s, 30° above the ground. The boundary rope is 75 m away. Is it a six? You can know before the ball lands, with a pencil and three lines of working.

That is what this unit is about: describing motion so exactly that you can predict it. **Kinematics** describes *how* things move: where they are, how fast, which way, and how that changes. Why they move is the next unit's business (Dynamics). Here you will learn to read the story of a journey from its graphs, to see motion from a moving bus, to derive the equations of motion three ways, and to work out the whole flight of a thrown stone, and so answer the batter's question (Exercise 36).

3.1 Introduction

The word **kinematics** comes from the Greek *kinesis*, motion, the same root as *cinema* (moving pictures) and *kinesiology* (the study of how the body moves). It is the study of **motion without asking about its causes**: no forces, no masses, only positions, velocities, accelerations and times.

A STORY FROM SCIENCE
Who named it, and who made it a science

The French physicist André-Marie Ampère, best known for electric current, coined the word *cinématique* in 1834, when he sorted all the sciences into a great table and felt that the description of motion, apart from its causes, deserved a name of its own.
In the 1800s, kinematics grew into the science of machines. The German engineer **Franz Reuleaux** (1829–1905), who taught at the Royal Technical Academy in Berlin and later led it, studied how the parts of a machine (links, gears, cams, cranks) guide one another's motion, and is called the "father of kinematics". He designed more than 800 teaching models of mechanisms. Most of those kept in Europe were destroyed in the Second World War; the largest collection left, 220 models, is at Cornell University in the United States.
A century later, the American engineer **Ferdinand Freudenstein** (1926–2006) used digital computers, then new, to *design* mechanisms: given the motion you want, find the linkage that makes it. The equation he found for a four-bar linkage is still taught, and he is called the "father of modern kinematics".

Kinematics is used wherever motion must be described exactly: the orbits of planets and moons in astronomy; engines, robot arms and cranes in engineering; the joints of the human skeleton in biomechanics; a satellite's track, a drone's flight path, a cricket ball's line in a TV replay.

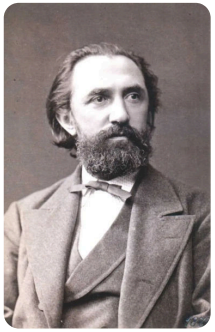


Figure 3.1 *Franz Reuleaux (1829–1905), who turned the study of machines' motion into a science.*
Photo: Unknown author, Public domain

Rest and motion are relative

A body is **in motion** if its position changes with time relative to its surroundings, and **at rest** if it does not. But "relative to its surroundings" matters:

- Two friends sitting in a moving bus are at rest relative to each other, and both are moving relative to the road.
- A tree in Bhaktapur is at rest relative to the ground, but seen from space it is carried round the Earth's axis at about 400 m/s, and round the Sun at about 30 km/s.

So **rest and motion are relative**: whenever you say how something moves, you must say what you measure it from, the **frame of reference**. Usually it is the ground.

Whether a body stays at rest is a question of forces, not kinematics, but one point is worth getting right now: a body with **no resultant force** on it is not necessarily at rest. It keeps whatever velocity it has: at rest if it was at rest, moving in a straight line at a steady speed if it was moving (Newton's first law, Unit 4).

Check 1
You sit in a bus moving north at 15 m/s. Relative to the driver, you are:

a at rest
b moving north at 15 m/s
c moving south at 15 m/s
d moving at 30 m/s

3.2 Distance, displacement, speed and velocity

Distance and displacement

Distance is the total length of the path a body has traveled. It depends on the shape of the path, and it is a scalar: it has a size but no direction, and it can never decrease.

Displacement is the change in position: the straight line from the starting point to the end point, **with its direction**. It is a vector (Unit 2). Its size is the shortest distance between the two points, whatever path was taken.

- An ant walks once round a circular plate of radius 10 cm. The **distance** it walks is the circumference, $2\pi \times 10 \approx 63$ cm. It ends where it started, so its **displacement is zero**.
- A ball thrown straight up 5 m falls back into your hand. Distance: 5 m up + 5 m down = 10 m. Displacement: zero.
- You go from P to Q along one zig-zag path and come back to P along another. The distance is the sum of the two path lengths; the displacement is zero. If you go only from P to Q, the size of the displacement is the straight-line distance PQ, whatever the zig-zags.
- So the size of the displacement is **less than or equal to** the distance. The two are equal only when a body moves in a straight line without turning back.

Check 2
A runner completes one and a half laps of a circular track of radius 50 m. The size of her displacement is:

a 0
b 50 m
c 100 m
d 471 m

Speed and velocity

Speed is the rate of change of distance: how much path is covered per unit time. It is a scalar, worked out with ordinary arithmetic:

$$\text{speed} = \frac{\text{distance}}{\text{time}}, \qquad v = \frac{d}{t}$$

Velocity is the rate of change of **displacement**. It is a vector: it has the speed's size **and** the direction of motion, and two velocities are combined by vector addition.

$$\text{velocity} = \frac{\text{displacement}}{\text{time}}$$

The SI unit of both is the meter per second, m/s.

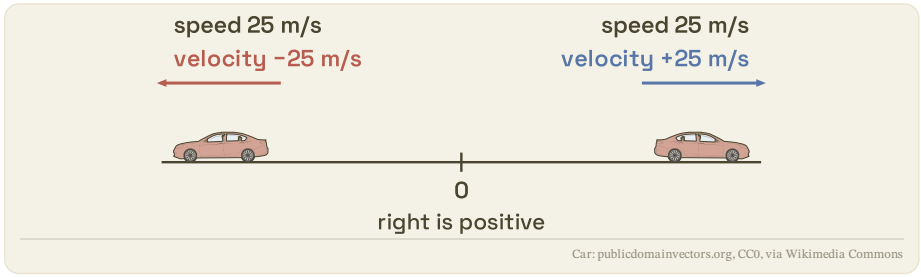


Figure 3.2 *Two cars drive away from the same point at 25 m/s in opposite directions. Their **speeds** are the same. Their **velocities** are not: 25 m/s to the right and 25 m/s to the left. Taking right as positive, they are +25 m/s and −25 m/s.*

In everyday talk the two words are mixed up. In physics, say **velocity** when the direction matters, and **speed** when only the size does.

- **Uniform velocity:** a body has uniform (constant) velocity if it makes **equal displacements in equal intervals of time**, however small the intervals. That means the same speed *and* the same direction: a straight line at a steady speed.

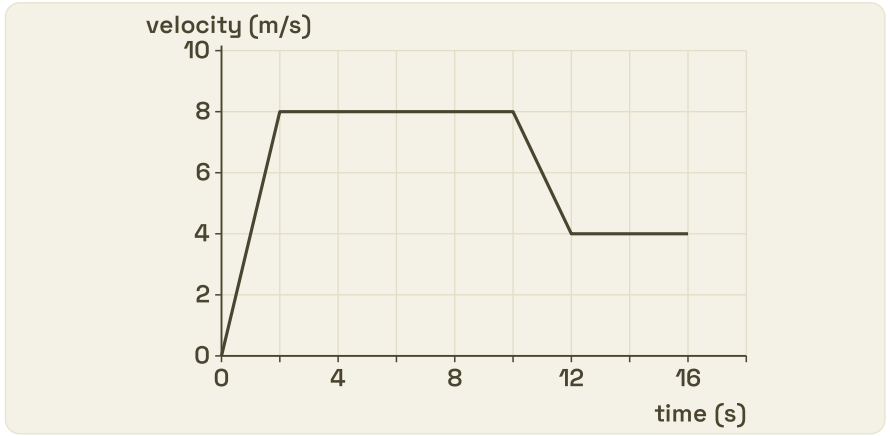


Figure 3.28 *For question 26.*

27. A child starts running from rest on a level road, speeding up uniformly for 8 s to 10 m/s. She keeps this speed for T s, then slows down uniformly until she stops. She runs 800 m in all, in 100 s.
    a. Sketch the velocity–time graph of her run.
    b. Find T.

28. The velocity–time graph of an object is shown below.
    a. Find its acceleration from $t = 0$ to $t = 4$ s.
    b. Find its deceleration from $t = 7$ s to $t = 10$ s.
    c. Find its total displacement from $t = 0$ to $t = 7$ s.

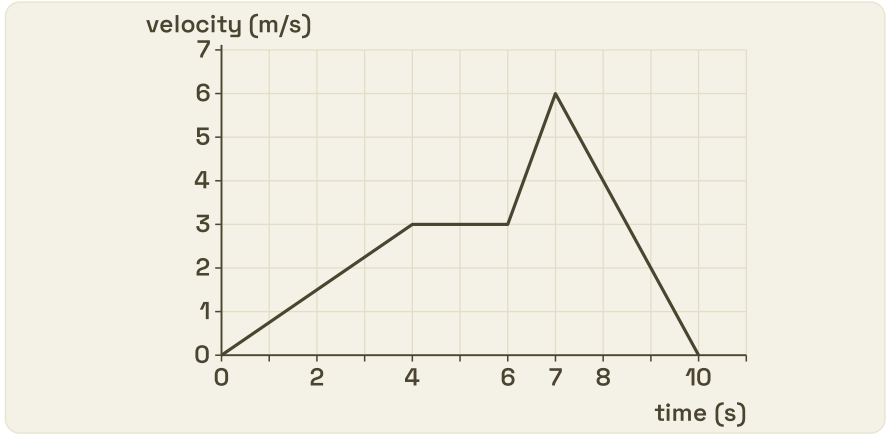


Figure 3.29 *For question 28.*

Numerical problems

29. A car moving at 10 m/s accelerates uniformly at 1 m/s² until it reaches 15 m/s. Find i. the time taken, ii. the distance covered while accelerating, and iii. its velocity 100 m from where it began to accelerate, if it keeps accelerating at 1 m/s².

30. A ball is thrown vertically upward at 20 m/s. Find i. the time it takes to return to the thrower and ii. the greatest height it reaches (g = 10 m/s²).

31. A ball is dropped from a height of 20 m and rebounds with ¾ of the speed with which it hits the ground. Find the time between the first and the second bounce (g = 10 m/s²).

Check 28

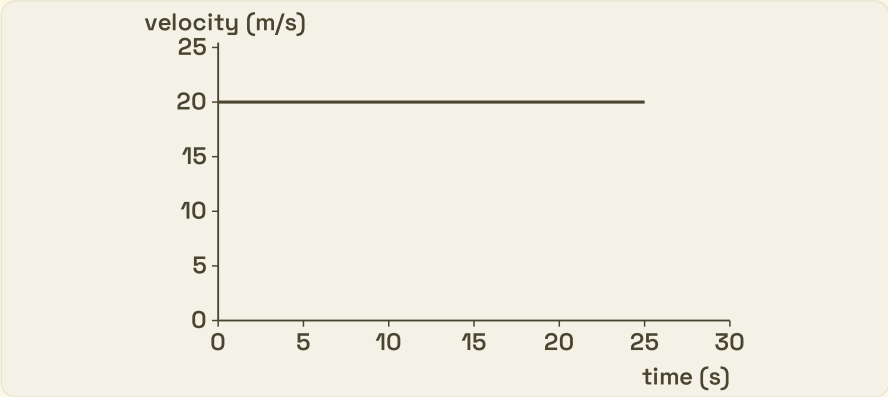


Figure 3.26 *For question 18: a velocity–time graph.*

18. From the velocity–time graph, the distance traveled by the body in 20 s is:

a 300 m
b 350 m
c 400 m
d 450 m

25. a. Show that the path of a projectile fired at an angle is a parabola.
b. Find the time of fall of a projectile fired horizontally from a height h.
c. At which point of a projectile's path are its velocity and acceleration at right angles?

26. The velocity–time graph of an object is shown below.
a. Find its acceleration from $t = 0$ to $t = 2$ s. If this is caused by a constant unbalanced force of 10 N, what is the object's mass?
b. Find its acceleration from $t = 10$ s to $t = 12$ s.
c. Find its total displacement from $t = 0$ to $t = 16$ s.

- **Variable velocity:** if either the speed or the direction changes, or both, the velocity is variable. A car going round a bend at a steady 40 km/h has a variable velocity.

Check 3

Which of these moves with uniform velocity?

a the tip of a fan blade turning steadily
b a stone falling from a roof
c a raindrop falling at its steady terminal speed in still air
d a car going round a roundabout at a steady 20 km/h

Average velocity

Average velocity is the total displacement divided by the total time taken:

$$v_{\text{av}} = \frac{\text{total displacement}}{\text{time taken}}$$

If a body is at position s_1 at time t_1 and at s_2 at time t_2,

$$v_{\text{av}} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{\Delta s}{\Delta t}$$

where Δ (delta) means "the change in". **Average speed**, in the same way, is the total distance divided by the total time.

WORKED EXAMPLE

Average of two speeds over equal distances

A man moves along a straight line with uniform velocity v_1 for a distance x, then with uniform velocity v_2 for the next equal distance x. Find his average velocity.

Answer

The times for the two halves are $t_1 = \frac{x}{v_1}$ and $t_2 = \frac{x}{v_2}$.

$$v_{\text{av}} = \frac{x + x}{t_1 + t_2} = \frac{2x}{\frac{x}{v_1} + \frac{x}{v_2}} = \frac{2}{\frac{1}{v_1} + \frac{1}{v_2}} = \frac{2v_1 v_2}{v_1 + v_2}$$

For example, 40 km/h out and 60 km/h back over the same road gives $\frac{2 \times 40 \times 60}{100} = 48$ km/h, **not** 50 km/h. He spends longer at the slower speed, so the slower speed counts for more.

CAREFUL

Not the average of the two speeds

$\frac{v_1 + v_2}{2}$ is the right answer only when the two speeds last for **equal times**. Over **equal distances**, the average is $\frac{2v_1 v_2}{v_1 + v_2}$ (the harmonic mean), which is always smaller. Exercise 1 is built on this trap.

YOUR TURN

Equal times instead

A cyclist rides for 1 hour at 12 km/h and then for 1 hour at 18 km/h along a straight road. Find her average speed.

Answer

Distance = 12 + 18 = 30 km in 2 h, so the average speed is 30 ÷ 2 = **15 km/h**. Here the plain average of the speeds works, because the two speeds last equal times.

Instantaneous velocity

A bus's speedometer doesn't show an average over the trip: it shows how fast the bus is going **now**. To find the velocity at one instant, take the average velocity over a smaller and smaller interval around that instant. The value it settles to as Δt shrinks towards zero is the **instantaneous velocity**:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

$\frac{ds}{dt}$ is read "d s by d t", the **derivative** of position with respect to time: the rate at which position is changing at that instant. If you know the position as a formula in t, you find the velocity by differentiating it. Two rules are enough for this unit:

- the derivative of a constant is 0;
- the derivative of t^n is $n\,t^{n-1}$ (so $t \to 1$, $t^2 \to 2t$, $t^3 \to 3t^2$, $t^4 \to 4t^3$), and a number multiplying it stays: $5t^2 \to 10t$.

WORKED EXAMPLE

Differentiating a position

A particle moves so that its position is $s = 3t^2 + 2t$ meters. Find its velocity at $t = 4$ s, and its average velocity over the first 4 s.

Answer

Instantaneous: $v = \dfrac{ds}{dt} = 6t + 2$. At $t = 4$ s, $v = 6 \times 4 + 2 = \mathbf{26\ m/s}$.

Average: $s(0) = 0$, $s(4) = 3 \times 16 + 8 = 56$ m, so $v_{\text{av}} = \dfrac{56 - 0}{4 - 0} = \mathbf{14\ m/s}$. It is less than 26 m/s because the particle was going slower earlier.

Position–time graphs

A **position–time** (displacement–time, or for straight-line motion distance–time) graph puts time across and position up. Its **slope** (gradient) is the velocity.

- Pick two points on the curve, (t_1, s_1) and (t_2, s_2). The straight line joining them (a **chord**) has slope $\dfrac{s_2 - s_1}{t_2 - t_1} = \dfrac{\Delta s}{\Delta t}$: the **average velocity** between those times.
- Bring the two points together and the chord becomes the **tangent**, the line that just touches the curve at one point. Its slope is $\dfrac{ds}{dt}$: the **instantaneous velocity** at that instant.

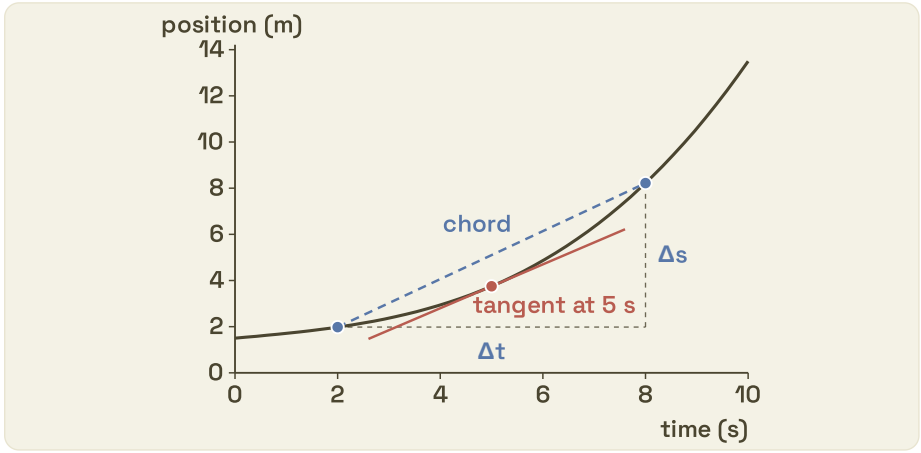


Figure 3.3 A position–time curve for a body speeding up. The chord between 2 s and 8 s (dashed) has slope Δs/Δt, the average velocity over those 6 s. The tangent at 5 s has slope ds/dt, the instantaneous velocity at 5 s. The curve is getting steeper, so the body is going faster and faster.

For **uniform velocity** the graph is a straight line (Figure 3.4): its slope is the same at every point, and the average and instantaneous velocities are equal. A curve means the velocity is changing.

Reading the shapes:

- **steeper** = faster;
- **flat** (horizontal) = not moving, at rest;
- sloping **down** = moving the other way, back towards the start (negative velocity);
- a straight line can never be **vertical**: that would be a change of position in no time, an infinite speed.

Notice that journey C starts at $t = 2$ s. The zero of time is a choice, like the zero of a ruler: an instant after it has a positive time, and an instant before it, if the story needed one, would have a negative time.

Check 26

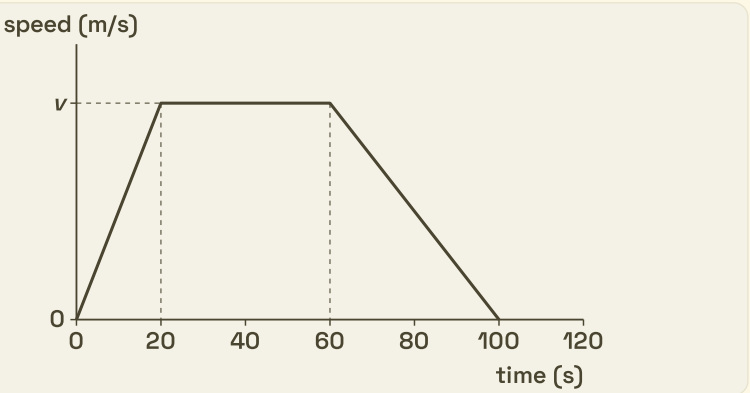


Figure 3.24 For question 16: a speed–time graph with an unknown top speed v.

16. The total displacement in the speed–time graph above is 1750 m. The value of v is:

a 15 m/s
b 20 m/s
c 25 m/s
d 28 m/s

Check 27

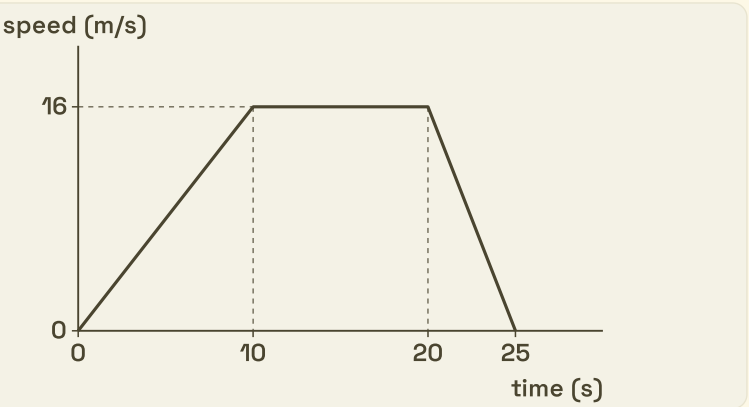


Figure 3.25 For question 17: a car's speed–time graph.

17. From the speed–time graph above, the car's deceleration in the last 5 s is:

a 3 m/s²
b 3.2 m/s²
c 1.6 m/s²
d 2.4 m/s²

Check 24



Figure 3.22 *For question 14: a car's displacement–time graph.*

14. From the car's displacement–time graph above, for how long does the car stay stopped?

a 10 s
b 20 s
c 30 s
d 15 s

Check 25

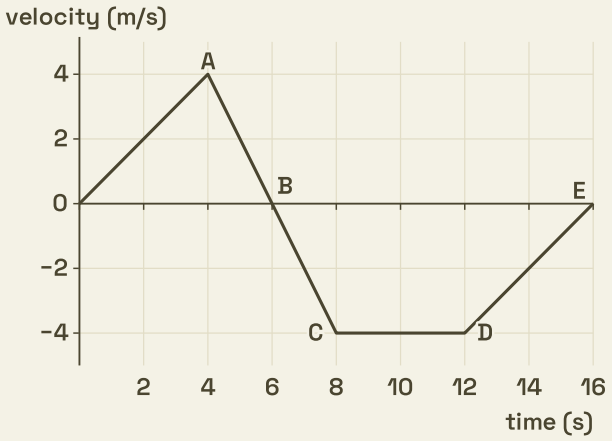


Figure 3.23 *For question 15: a velocity–time graph that goes below the time axis.*

15. From the velocity–time graph above, the total displacement of the particle from 0 to 16 s is:

a 8 m
b −16 m
c 16 m
d −8 m

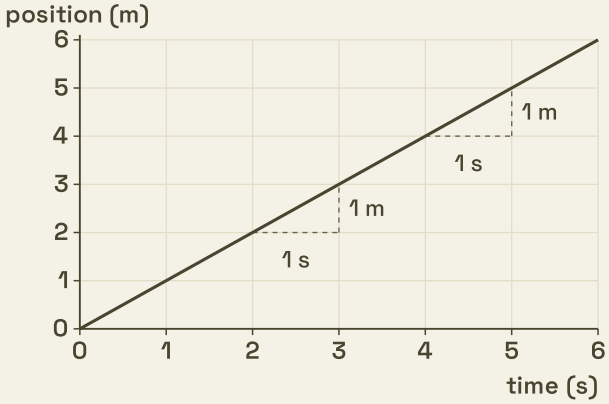


Figure 3.4 *Uniform velocity: equal displacements in equal times give a straight line. Its slope, 1 m every second here, is the same everywhere.*

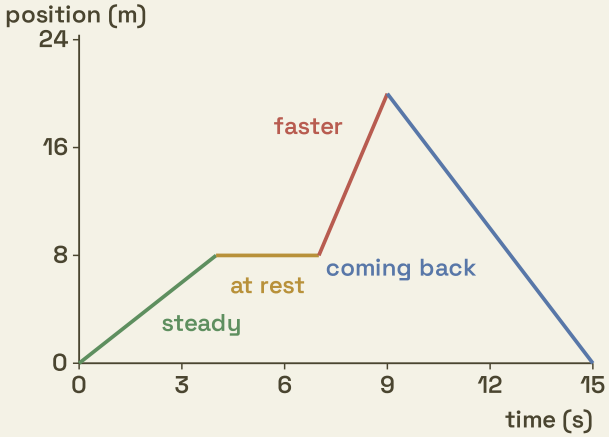


Figure 3.5 *One journey in four stages: uniform speed away from the start; at rest; a faster uniform speed away; then a steady speed back to the start (a negative slope), arriving where it began.*

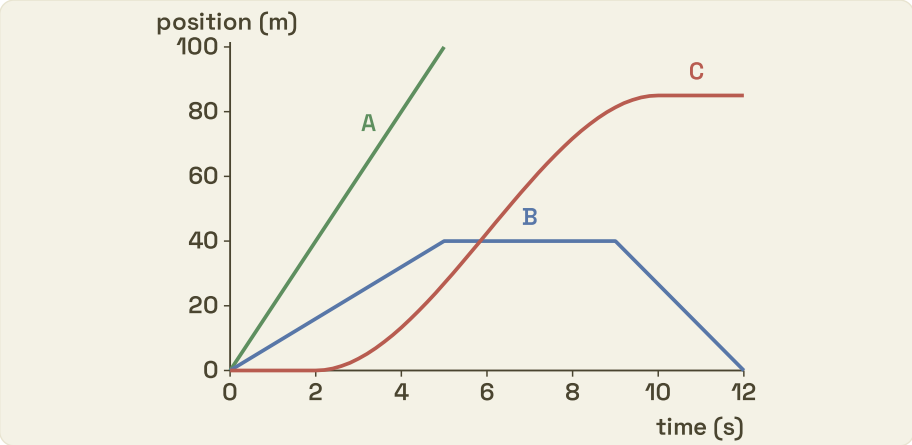


Figure 3.6 Three journeys on one graph. **A** (green) moves at a fast, steady 20 m/s, 100 m in 5 s. **B** (blue) goes at a steady 8 m/s for 5 s, stops for 4 s, and comes back to the start at 40 ÷ 3 ≈ 13 m/s. **C** (red) starts 2 s after the others, speeds up (the curve gets steeper), slows down (the curve flattens), and stops 85 m away.

Check 4

On a position–time graph, a body's line is horizontal from 4 s to 9 s. Between those times the body is:

a moving at a steady speed
b at rest
c accelerating
d moving backwards

3.3 Acceleration and velocity–time graphs

What acceleration is

Throw a ball straight up at 50 m/s. Ignoring air resistance, and taking g as 10 m/s² to keep the numbers round, its velocity goes like this:

Time (s)	0	1	2	3	4	5	6	7	8	9	10
Velocity (m/s, up +)	50	40	30	20	10	0	−10	−20	−30	−40	−50

Every second the velocity changes by the same amount, 10 m/s downward: it falls by 10 m/s on the way up, stops for an instant at the top after 5 s, and grows by 10 m/s each second on the way down, reaching the thrower at 50 m/s after another 5 s.

That change of velocity per unit time is the **acceleration**: the rate of change of velocity. It is a vector, in m/s². When a body is slowing down, its acceleration points against its velocity, and we call it a **deceleration** or **retardation** (a negative acceleration, if the velocity is taken as positive).

- **Uniform acceleration:** equal changes of velocity in equal intervals of time (the ball above, or a stone falling).
- **Variable (non-uniform) acceleration:** unequal changes in equal intervals (a bus in traffic).
- **Average acceleration:** the total change in velocity divided by the total time. If the velocity is v_1 at t_1 and v_2 at t_2,

$$a_{\text{av}} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

- **Instantaneous acceleration:** the acceleration at one instant, the limit of the average acceleration as the interval shrinks to zero:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

For **uniform** acceleration, the acceleration is the same at every instant, so it equals the average over any interval: a body going from u to v in time t has $a = \dfrac{v - u}{t}$.

Check 19

9. The simplest type of motion, along a straight-line path, is:

a plane motion
b rectilinear motion
c curvilinear motion
d circular motion

Check 20

10. A train covers 90 miles between 1 p.m. and 4 p.m. at a steady speed. How fast was it going at 2 p.m.?

The textbook gives only the 90 miles in 3 hours, without "at a steady speed", and an average speed can't tell the speed at one instant (the train could have been standing at a station at 2 p.m.), so the book's numbers don't work as asked. With a steady speed they do.

a 90 mph
b 30 mph
c 40 mph
d 80 mph

Check 21

11. A police officer on a planet with no air drops a pair of handcuffs and a handkerchief from the same height at the same time. Which reaches the ground first?

a the handkerchief
b the handcuffs
c it depends on how high they were dropped from
d they both reach the ground at the same time

Check 22

12. A ball dropped from a tower covers half the total height in the last second of its fall. Taking $g = 10$ m/s², the total time of fall is:

a $(1 + \sqrt{2})$ s
b $(2 + \sqrt{2})$ s
c $2\sqrt{2}$ s
d $\sqrt{2}$ s

Check 23

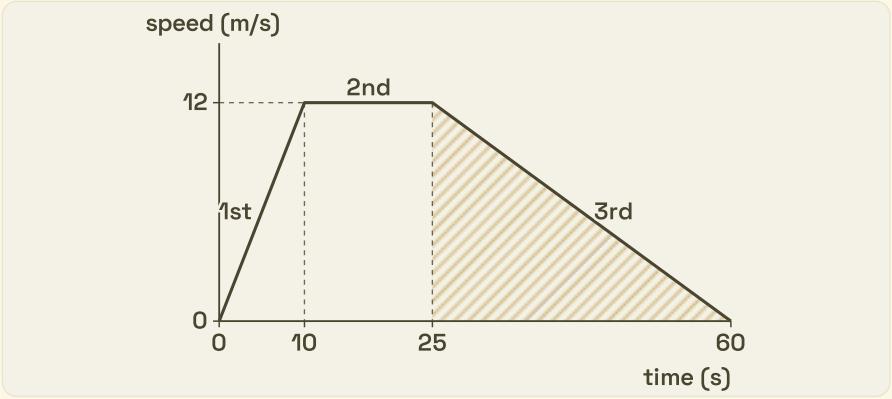


Figure 3.21 For question 13: a speed–time graph in three stages.

13. From the speed–time graph above, the distance traveled by the particle in the third stage is:

a 220 m
b 200 m
c 105 m
d 210 m

Exercise

Choose the correct option

Check 11

1. A bus travels ⅓ of a distance at 10 km/h, the next ⅓ at 20 km/h and the last ⅓ at 60 km/h. Its average speed is:

a 9 km/h b 16 km/h
c 18 km/h d 48 km/h

Check 12

2. The velocity of a particle moving with uniform acceleration is 10 m/s at one instant and 20 m/s 5 s later. What was its velocity 3 s before it reached 10 m/s?

a 5 m/s b 4 m/s
c 6 m/s d 8 m/s

Check 13

3. A particle starts from rest with a constant acceleration for 20 s. It travels S_1 in the first 10 s and S_2 in the next 10 s. Then:

a $S_1 = S_2$ b $S_2 = 2S_1$
c $S_2 = 3S_1$ d $S_1 = 4S_2$

Check 14

4. The displacement of a particle is proportional to the first power of time. Its acceleration is:

a a small finite value b zero
c infinite d a large finite value

Check 15

5. In one hour, the tip of the minute hand of a clock has:

a zero displacement b zero distance
c zero average speed d a non-zero average velocity

Check 16

6. A bullet fired into a wooden block loses half its velocity after going 40 cm into it. How much farther does it go before stopping (assuming a uniform retardation)?

The textbook prints the options in meters (4000/3 m, 2200/3 m, 2200/5 m, 2000/3 m), and none of them fits: the answer is 40/3 cm, so the book's options don't work. The options below are corrected.

a 40/3 cm b 20/3 cm
c 40 cm d 80/3 cm

Check 17

7. A cyclist on a circular track of radius 25 m completes half a lap in 25 s. Her average velocity is:

a 4 m/s b 2 m/s
c zero d π m/s

Check 18

8. Two railway tracks run side by side from north to south. Train A runs from north to south at 54 km/h, and train B from south to north at 90 km/h. The velocity of A relative to B is:

a 20 m/s b 30 m/s
c 40 m/s d 50 m/s

Answer these questions

19. a. What are average speed and average velocity?

WORKED EXAMPLE

Differentiating twice

The position of a particle is $x = 5 + 4t + 2t^2$ (meters, seconds). Find its instantaneous velocity and acceleration.

Answer

Velocity: $v = \dfrac{dx}{dt} = 0 + 4 + 4t = 4 + 4t$ m/s.

Acceleration: $a = \dfrac{dv}{dt} = \mathbf{4\ m/s^2}$, the same at every instant: a uniform acceleration.

What acceleration does not tell you

Acceleration and velocity are separate quantities, and their directions can differ:

- **Velocity changing direction, acceleration constant.** The ball thrown up: its velocity is up, then zero, then down, but its acceleration is 10 m/s² **downward** the whole time, even at the top, where the velocity is zero for an instant.
- **Velocity one way, acceleration the other.** A pendulum bob swinging east of the middle is slowed by a pull back towards the middle: eastward velocity, westward acceleration. A bus braking while moving north is the same.
- **Constant speed, changing velocity.** A body going round a circle at a steady speed keeps changing direction, so its velocity changes and it is accelerating (towards the center, Unit 6).
- **Constant acceleration, changing speed.** This is the most common case of all: a falling stone, a ball in flight. Its acceleration is a steady g downward while its speed keeps changing.

Check 5

A car moves east and its speed is decreasing. The direction of its acceleration is:

a east b west
c zero, since it is moving d up

Velocity–time graphs

A **velocity–time graph** tells the story of a journey in a straight line: time across, velocity up. It holds two quantities at once:

- the **slope** of the graph is the **acceleration** ($a = \dfrac{dv}{dt}$);
- the **area** between the graph and the time axis is the **displacement** (area above the axis positive, area below negative).

Why the area? Over a short time Δt the velocity is nearly constant, so the displacement is $v\,\Delta t$, a thin strip of height v and width Δt. Add up all the strips and you get the area under the graph. (In calculus that sum is the integral, $s = \int v\,dt$.)

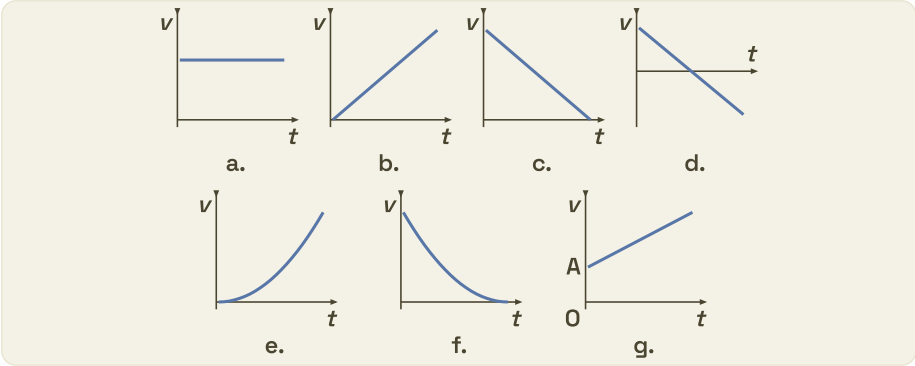


Figure 3.7 The common shapes of a velocity–time graph. a. Constant velocity. b. Constant acceleration from rest. c. Constant deceleration to rest. d. A ball thrown up: the velocity falls through zero (at the top) and goes on to negative (coming down), along one straight line. e. Increasing acceleration (the curve gets steeper). f. Non-uniform retardation. g. Uniform acceleration with an initial velocity OA = u.

- A **horizontal** line: constant velocity, zero acceleration.
- A straight line sloping **up**: uniform acceleration. Sloping **down**: uniform deceleration.
- A **curve**: the acceleration is changing. Getting steeper: increasing acceleration; flattening: decreasing.
- A line **crossing the time axis**: the body stops for an instant and reverses.

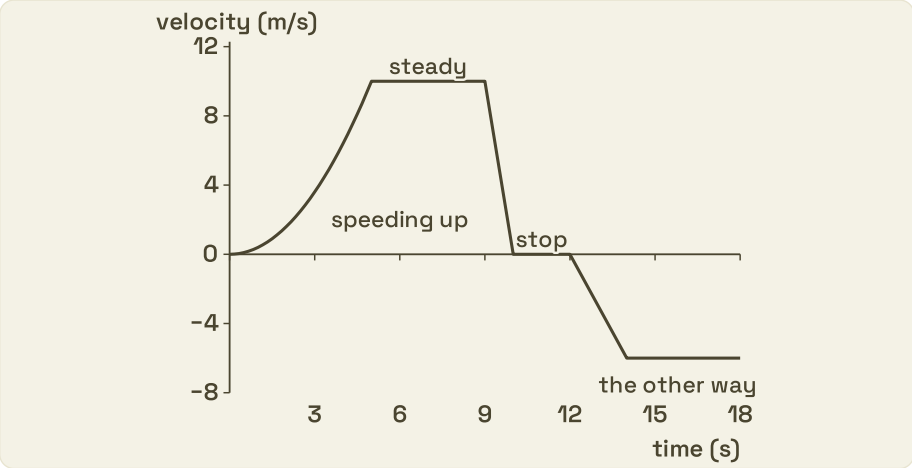


Figure 3.8 *One journey's velocity–time graph: speeding up slowly, then quickly, a steady speed, a rapid deceleration to a stop, a pause, and then speeding up and holding a steady speed in the opposite direction (negative velocity).*

Rules every velocity–time graph obeys

1. It is **one connected line, with one value at each instant**: a body can't have two velocities at once, or jump from one velocity to another in no time.
2. It always runs **forward in time**: it can never turn back towards the left.
3. The **area under it** between two times is the displacement in that interval (counting area below the axis as negative). The total distance is the sum of the areas without their signs.
4. It is **never vertical**: a vertical line would be a change of velocity in zero time, an infinite acceleration.
5. Uniformly increasing velocity is a straight line of positive slope; uniformly decreasing velocity, a straight line of negative slope.
6. When the acceleration itself increases (or decreases), the line is a curve whose slope keeps increasing (or decreasing).

Draw your own journey below and watch both of the graph's numbers at once.

Check 6

A velocity–time graph is a straight line from 20 m/s at 0 s to 0 m/s at 4 s. The body's acceleration and the distance it covers are:

a -5 m/s² and 80 m
b -5 m/s² and 40 m
c 5 m/s² and 40 m
d -4 m/s² and 40 m

WORKED EXAMPLE

A suitcase falls from a climbing plane

An airplane is climbing at 90.0 m/s at 23.0° above the horizontal. When it is 114 m directly above a dog standing on level ground, a suitcase falls out of the luggage compartment. How far from the dog does it land? Ignore air resistance and take $g = 10$ m/s².

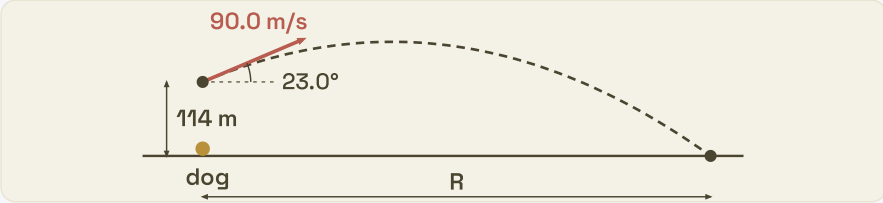


Figure 3.20 *The suitcase leaves the plane 114 m above the dog with the plane's velocity, 90.0 m/s at 23.0° above the horizontal, so it first rises and then falls, landing R beyond the dog.*

Answer

At the moment it falls out, the suitcase has the **plane's velocity**: it is a projectile fired upward at 23.0°, from 114 m up. $u_x = 90.0\cos 23.0^\circ = 82.8$ m/s; $u_y = 90.0\sin 23.0^\circ = 35.2$ m/s (upward).

Time of flight. Taking up as positive, it lands when its displacement is $y = -114$ m:

$$-114 = 35.2\,T - \tfrac{1}{2}\times 10\times T^2 \quad\Rightarrow\quad 5T^2 - 35.2\,T - 114 = 0$$

$$T = \frac{35.2 + \sqrt{35.2^2 + 4\times 5\times 114}}{2\times 5} = \frac{35.2 + 59.3}{10} = 9.45\ \text{s}$$

(the other root is negative, before it fell out).

Range: $R = u_x T = 82.8 \times 9.45 = \mathbf{783\ m}$ beyond the dog. (With $g = 9.8$ m/s², $T = 9.60$ s and $R = 795$ m.)

DID YOU KNOW?

Why javelins and long jumpers don't use 45°

The 45° rule assumes a launch from ground level, no air, and a speed that doesn't depend on the angle. Real throws break all three. A javelin leaves the hand about 2 m above the ground and is shaped to glide, and an athlete can throw faster at a flatter angle; the best javelin release angles are about 32–36°. A long jumper loses speed if she tries to leap steeply off a running start: elite jumpers take off at only about 20°, trading angle for speed.

IN SHORT

Key ideas of Unit 3

- **Distance** (path length, scalar) and **displacement** (change of position, vector); **speed** = distance ÷ time, **velocity** = displacement ÷ time.
- Instantaneous velocity $v = \dfrac{ds}{dt}$ (slope of the position–time graph); acceleration $a = \dfrac{dv}{dt}$ (slope of the velocity–time graph); the area under a velocity–time graph is the displacement.
- Relative velocity: $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$; bring the observer to rest by reversing his velocity; at an angle, $v = \sqrt{a^2 + b^2 - 2ab\cos\theta}$. Rain on a walker: $\tan\theta = v_p/v_r$.
- For uniform acceleration: $v = u + at$, $s = ut + \frac{1}{2}at^2$, $v^2 = u^2 + 2as$, $s = \frac{1}{2}(u+v)t$, $s_{n\text{th}} = u + \frac{a}{2}(2n-1)$; under gravity, $a = g$.
- Horizontal launch from height h: $y = \dfrac{g}{2u^2}x^2$, $T = \sqrt{2h/g}$, $R = u\sqrt{2h/g}$.
- Launch at θ: $y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$, $T = \dfrac{2u\sin\theta}{g}$, $H = \dfrac{u^2\sin^2\theta}{2g}$, $R = \dfrac{u^2\sin 2\theta}{g}$; greatest range u^2/g at 45°; θ and 90° $-\,\theta$ give the same range.

Two angles, one range

Fire at θ and at $90^\circ - \theta$ with the same speed:

$$R_{90^\circ-\theta} = \frac{u^2 \sin(180^\circ - 2\theta)}{g} = \frac{u^2 \sin 2\theta}{g} = R_\theta$$

since $\sin(180^\circ - x) = \sin x$. So **complementary angles** (adding to 90°) give the **same range**, with different paths: the higher one stays up longer and climbs higher. 30° and 60° land at the same spot; so do 18° and 72°, or 20° and 70°.

Launch your own projectiles and see the 45° maximum and the complementary pairs for yourself.

Velocity at any instant

The horizontal part stays $v_x = u\cos\theta$; the vertical part is $v_y = u\sin\theta - gt$. So

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}, \qquad \tan\alpha = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta}$$

where α is the angle of the velocity above the horizontal (negative on the way down). At the top, $v_y = 0$: the velocity is horizontal, $u\cos\theta$, at right angles to the downward acceleration g.

Check 10

A ball is kicked at 20 m/s at 30° above the ground (g = 10 m/s²). At the top of its flight, its speed is:

a 0 b 10 m/s
c 17.3 m/s d 20 m/s

WORKED EXAMPLE

A shell fired at 30°

A shell is fired from the ground at 500 m/s at 30° to the horizontal. Ignoring air resistance (and taking $g = 10$ m/s²), find its horizontal range, its greatest height, and the time it takes to reach the greatest height. What is the least speed that would give the same range?

Answer

Range: $R = \dfrac{u^2\sin 2\theta}{g} = \dfrac{500^2 \times \sin 60^\circ}{10} = \dfrac{250\,000 \times 0.866}{10} =$ **21 651 m**, about 21.7 km.

Greatest height: $H = \dfrac{u^2\sin^2\theta}{2g} = \dfrac{250\,000 \times 0.5^2}{20} =$ **3125 m**.

Time to the top: $\dfrac{u\sin\theta}{g} = \dfrac{500 \times 0.5}{10} =$ **25 s**.

Least speed for the same range: fire at 45°, where $R = \dfrac{u'^2}{g}$: $u' = \sqrt{Rg} = \sqrt{21\,651 \times 10} =$ **465 m/s**.

(A real shell at these speeds loses much of its range to air resistance; the numbers show the ideal case.)

WORKED EXAMPLE

The cyclist and the athlete

As a cyclist starts from rest, an athlete running at a steady 20 m/s passes her. The cyclist speeds up uniformly to 30 m/s in 6 s and then keeps that speed (Figure 3.9). Find:

a. how far ahead the athlete is after 6 s;
b. when the cyclist catches up with the athlete;
c. how far from the start they are at that instant.

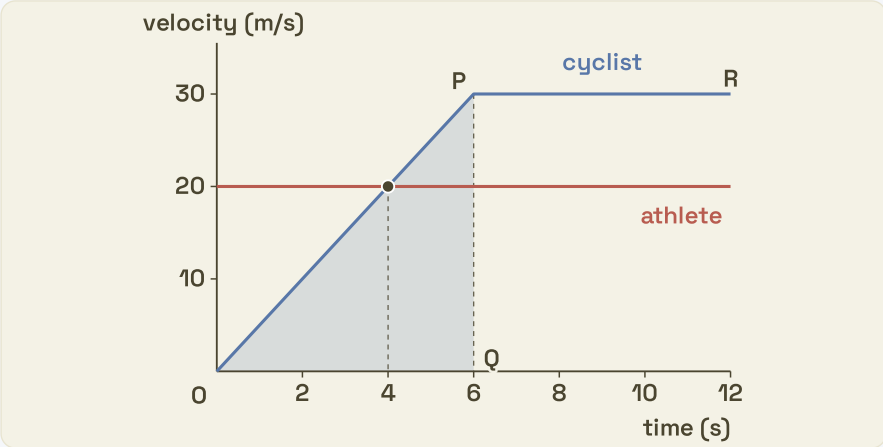


Figure 3.9 The cyclist (blue, OPR) speeds up from rest to 30 m/s in 6 s, then rides steadily; the athlete (red) runs at a steady 20 m/s. The areas under the two lines are the distances they cover.

Answer

a. The cyclist's distance in 6 s is the area of triangle OPQ: $\frac{1}{2} \times 6 \times 30 = 90$ m. The athlete's is the rectangle $20 \times 6 = 120$ m. The athlete is **120 − 90 = 30 m ahead**.
b. From then on the cyclist gains on the athlete at $30 - 20 = 10$ m/s (her velocity relative to the athlete, Section 3.4). She closes the 30 m gap in $30 \div 10 = 3$ s, so she catches up **9 s** after starting.
c. In 9 s the athlete runs $20 \times 9 =$ **180 m**. Check with the cyclist: $90 + 30 \times 3 = 180$ m. They are side by side, 180 m from the start.

Notice that the lines cross at 4 s, when the two have the **same velocity**, not when the cyclist catches up. At that moment the athlete is farthest ahead; after it, the gap starts to close.

3.4 Relative velocity

How fast something seems to move depends on how the observer moves:

- Run through rain, and the rain seems to come slanting at your face.
- Two buses at the same speed, passing each other in opposite directions, each seem to the passengers of the other to rush past at twice that speed.
- An airplane seen from a moving bus seems to move differently from how it moves seen from the ground.

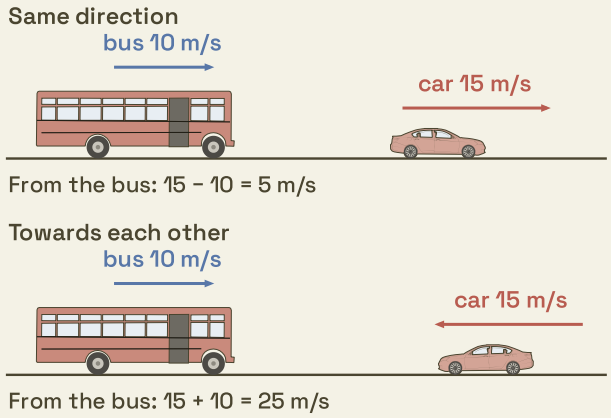
The velocity of a body **as measured by an observer who is himself moving** is its **relative velocity**. The trick for finding it is always the same: **bring the observer to rest** in your imagination, by giving both bodies the observer's velocity reversed. What is left is the velocity the observer sees.

Rules for relative velocity

If bodies A and B have velocities $\vec{v}_A$ and $\vec{v}_B$ relative to the ground, the velocity of A **relative to B** is

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

and the velocity of B relative to A is $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = -\vec{v}_{AB}$: the same size, the opposite direction.



Bus: traced from a photo by Saiphani02, CC BY 4.0, via Wikimedia Commons
Car: publicdomainvectors.org, CC0, via Wikimedia Commons

Figure 3.10 *A car at 15 m/s overtakes a bus at 10 m/s on the same road. To the bus's passengers, the car creeps past at 15 − 10 = 5 m/s. If the car met the bus coming the other way, they would see it rush past at 15 + 10 = 25 m/s.*

- **Same direction** (along a line): the sizes subtract. Car 15 m/s, bus 10 m/s the same way: $v_{\text{car,bus}} = 15 - 10 = 5$ m/s.
- **Opposite directions:** take one direction as positive, so the other velocity is negative, and the sizes **add**: $v_{AB} = v_A - (-v_B) = v_A + v_B$. Car 15 m/s, bus 10 m/s the other way: 25 m/s.
- **At an angle:** subtract the vectors. Reverse $\vec{v}_B$ to get $-\vec{v}_B$, and add it to $\vec{v}_A$ by the parallelogram (or triangle) law of Unit 2.

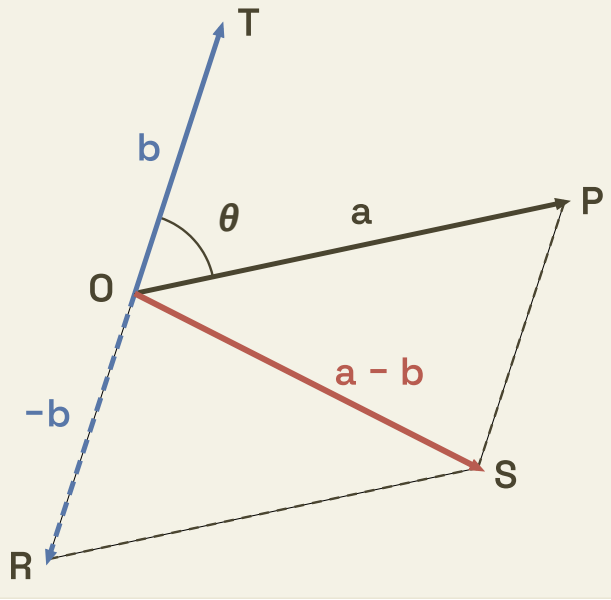


Figure 3.11 *Finding the velocity of A relative to B. OP is a, A's velocity; OT is b, B's velocity, at an angle θ to a. Reversing b gives OR = −b. The diagonal OS of the parallelogram on a and −b is a − b, A's velocity as B sees it.*

The angle between $\vec{a}$ and $-\vec{b}$ is $180^\circ - \theta$, so by the parallelogram law

$$|\vec{a} - \vec{b}| = \sqrt{a^2 + b^2 + 2ab\cos(180^\circ - \theta)} = \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2 \tag{3.13}$$

For a given launch, $\tan\theta$ and $\frac{g}{2u^2\cos^2\theta}$ are constants, so this is $y = bx - cx^2$: a **parabola** opening downward. The path of a projectile fired at an angle is parabolic too.

Time of flight T. It lands when $y = 0$ again:

$$0 = (u\sin\theta)\,T - \tfrac{1}{2}gT^2 = T\left(u\sin\theta - \tfrac{1}{2}gT\right)$$

$T = 0$ is the launch, so the landing is at

$$T = \frac{2u\sin\theta}{g}$$

Another way: at the top (A) the vertical velocity is zero. From $v_y = u_y - gt$, $0 = u\sin\theta - gt$, so it reaches the top at $t = \frac{u\sin\theta}{g}$. The way down takes as long as the way up (the path is symmetrical), so $T = \frac{2u\sin\theta}{g}$.

Maximum height H. At the top $v_y = 0$. From $v_y^2 = u_y^2 - 2gh$:

$$0 = (u\sin\theta)^2 - 2gH \quad\Rightarrow\quad H = \frac{u^2\sin^2\theta}{2g}$$

- Thrown **straight up** ($\theta = 90°$), $H = \frac{u^2}{2g}$: it doesn't depend on the mass, so a light and a heavy ball thrown up at the same speed reach the same height.
- How high and how far a body goes depends only on the launch **speed** and **angle** (and g). That is why a long jumper works on both her run-up speed and her take-off angle.

Horizontal range R:

$$R = u_x T = u\cos\theta\cdot\frac{2u\sin\theta}{g} = \frac{u^2\cdot 2\sin\theta\cos\theta}{g}$$

and since $2\sin\theta\cos\theta = \sin 2\theta$,

$$R = \frac{u^2\sin 2\theta}{g}$$

The greatest range: 45°

For a given speed, R is greatest when $\sin 2\theta$ is greatest, which is 1, when $2\theta = 90^\circ$:

$$\theta = 45^\circ, \qquad R_{\max} = \frac{u^2}{g}$$

So to throw farthest at a given speed, or to reach a given distance with the **least speed**, launch at **45°**.

Two useful comparisons:

- Thrown straight up, the greatest height a person can reach is $\frac{u^2}{2g}$. So $R_{\max} = \frac{u^2}{g}$ is **twice** that: if you can throw a stone 15 m high, you can throw it at most 30 m far.
- For the **same 45° flight**, the height reached is $H = \frac{u^2\sin^2 45^\circ}{2g} = \frac{u^2}{4g}$, so $R_{\max} = 4H$: at 45°, a projectile goes four times as far as it rises.

CAREFUL

Which H?

"$R_{\max} = 2H$" and "$R_{\max} = 4H$" are both true, about different heights. $2H$ compares the 45° range with the height of a throw **straight up** at the same speed; $4H$ compares it with the height of the **same 45° flight**. Say which one you mean.

YOUR TURN

Relate the ranges at 15° and 45°

At the same speed, how does the range at 15° compare with the range at 45°?

Answer

$R_{15} = \frac{u^2\sin 30^\circ}{g} = \frac{u^2}{2g}$ and $R_{45} = \frac{u^2\sin 90^\circ}{g} = \frac{u^2}{g}$, so $R_{15} = \frac{R_{45}}{2}$: half as far.

Fire a body horizontally with velocity u from a point O at height h. Take O as the origin, x horizontal and y measured **downward**. After a time t it is at P(x, y).

Horizontally ($u_x = u$, no acceleration):

$$x = ut \quad \Rightarrow \quad t = \frac{x}{u} \tag{3.10}$$

Vertically ($u_y = 0$, acceleration g downward):

$$y = u_y t + \tfrac{1}{2} g t^2 = \tfrac{1}{2} g t^2$$

The path. Put $t = \dfrac{x}{u}$ into the second:

$$y = \tfrac{1}{2} g \left(\frac{x}{u}\right)^2 = \left(\frac{g}{2u^2}\right) x^2 \tag{3.11}$$

g and u are constants, so this is $y = kx^2$, the equation of a **parabola**: the path is parabolic.

Time of flight T, the time it stays in the air: it lands when it has fallen h, so

$$h = \tfrac{1}{2} g T^2 \quad \Rightarrow \quad T = \sqrt{\frac{2h}{g}}$$

This is exactly the time a body **dropped** from the same height takes. So a body fired horizontally and one dropped at the same moment from the same height **land together**, whatever the horizontal speed.

Horizontal range R, the horizontal distance traveled before it lands:

$$R = uT = u\sqrt{\frac{2h}{g}}$$

Velocity at any instant. The horizontal part stays $v_x = u$; the vertical part grows as $v_y = u_y + gt = gt$. So

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + g^2 t^2}, \qquad \tan\alpha = \frac{v_y}{v_x} = \frac{gt}{u}$$

where α is the angle of the velocity **below the horizontal**. It lands with $v = \sqrt{u^2 + 2gh}$, since $v_y^2 = 2gh$ when it has fallen h.

Check 9

A stone is thrown horizontally at 15 m/s from a cliff 20 m high (g = 10 m/s²). It lands:

a 15 m from the foot of the cliff, after 1 s
b 30 m from the foot of the cliff, after 2 s
c 60 m from the foot of the cliff, after 4 s
d 30 m from the foot of the cliff, after 4 s

A projectile fired at an angle

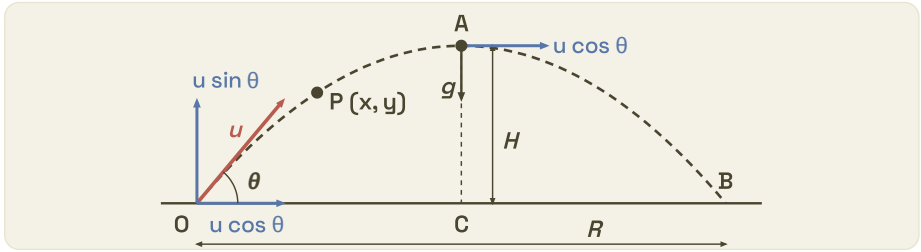


Figure 3.19 *A projectile fired from O at speed u, at an angle θ above the horizontal: its velocity has parts u cos θ across and u sin θ up. At P, after a time t, it is x across and y up. It reaches its greatest height at A, above C, the middle of the range, and lands at B, a range R from O.*

Fire a body from O at speed u, at an angle θ above the horizontal. Resolve the velocity (Unit 2):

$$u_x = u\cos\theta \text{ (horizontal)}, \qquad u_y = u\sin\theta \text{ (vertical, upward)}$$

Take y **upward** now, so the vertical acceleration is $-g$. After a time t the body is at P(x, y):

$$x = u_x t = (u\cos\theta)\, t \quad \Rightarrow \quad t = \frac{x}{u\cos\theta} \tag{3.12}$$
$$y = u_y t - \tfrac{1}{2} g t^2 = (u\sin\theta)\, t - \tfrac{1}{2} g t^2$$

The path. Substitute t:

$$y = u\sin\theta \cdot \frac{x}{u\cos\theta} - \tfrac{1}{2} g \left(\frac{x}{u\cos\theta}\right)^2$$

Two cases check the formula:

- $\theta = 0$ (same direction): $\sqrt{a^2 + b^2 - 2ab} = |a - b|$, the difference.
- $\theta = 180^\circ$ (opposite directions): $\sqrt{a^2 + b^2 + 2ab} = a + b$, the sum.

Check 7

Two trains on parallel tracks: A runs north at 20 m/s, B south at 15 m/s. The velocity of B relative to A is:

a 5 m/s south
b 5 m/s north
c 35 m/s south
d 35 m/s north

A swimmer in a river

A swimmer's velocity relative to the water, $\vec{v}_{pw}$ ("person–water"), and the water's velocity relative to the ground, $\vec{v}_{wg}$, add to give the swimmer's velocity relative to the ground:

$$\vec{v}_{pg} = \vec{v}_{pw} + \vec{v}_{wg}, \qquad \text{so} \qquad \vec{v}_{pw} = \vec{v}_{pg} - \vec{v}_{wg}$$

- Swimming **downstream**, the speeds add: $v_{pg} = v_{pw} + v_{wg}$.
- Swimming **upstream**, they subtract: $v_{pg} = v_{pw} - v_{wg}$. If the river is faster than the swimmer, he is carried backwards while swimming hard.

YOUR TURN

Rafting on the Trishuli

A raft is paddled at 2 m/s relative to the water. The Trishuli flows at 3 m/s. How fast does the raft move past the bank when paddled downstream, and when paddled upstream?

Answer

Downstream: $2 + 3 =$ **5 m/s** downstream. Upstream: $2 - 3 = -1$ m/s, so the raft still goes **1 m/s downstream**, backwards for the rafters, however hard they paddle.

Rain and a moving person

Rain falls straight down at v_r. To someone standing still, it falls vertically. Now walk at v_p. To find the rain's velocity relative to you, bring yourself to rest: give the rain your velocity reversed, $-\vec{v}_p$. The rain's velocity relative to you is the resultant of $\vec{v}_r$ (down) and $-\vec{v}_p$ (horizontal, towards you), at right angles:

$$v = \sqrt{v_r^2 + v_p^2}, \qquad \tan\theta = \frac{v_p}{v_r}$$

where θ is the angle the rain seems to make with the vertical. It slants **towards your face**, so hold your umbrella **tilted forward** by θ.

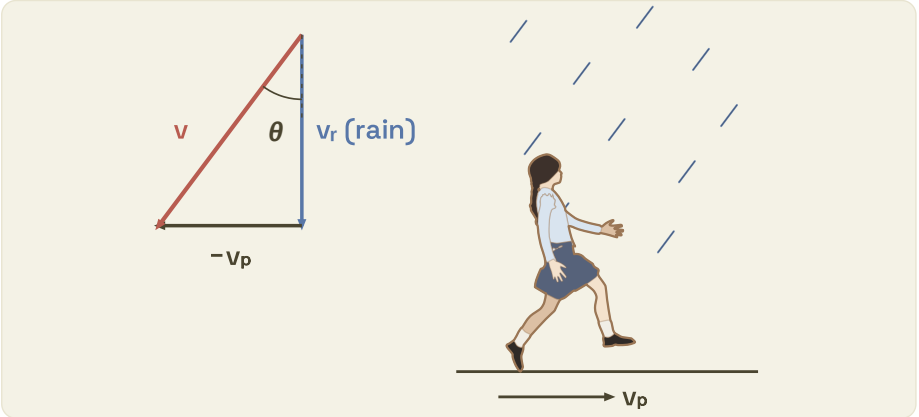


Figure 3.12 *Vertical rain seen by a person walking to the right at v_p. Adding $-v_p$ (reversed) to the rain's velocity v_r gives the rain's velocity relative to the person, v, slanting towards the person at θ to the vertical, with $\tan\theta = v_p/v_r$.*

The same idea works the other way. If rain is falling slantwise, towards the south-east, a man walking east at just the right speed, equal to the rain's eastward part, sees it fall **vertically**: his own velocity cancels the rain's sideways part.

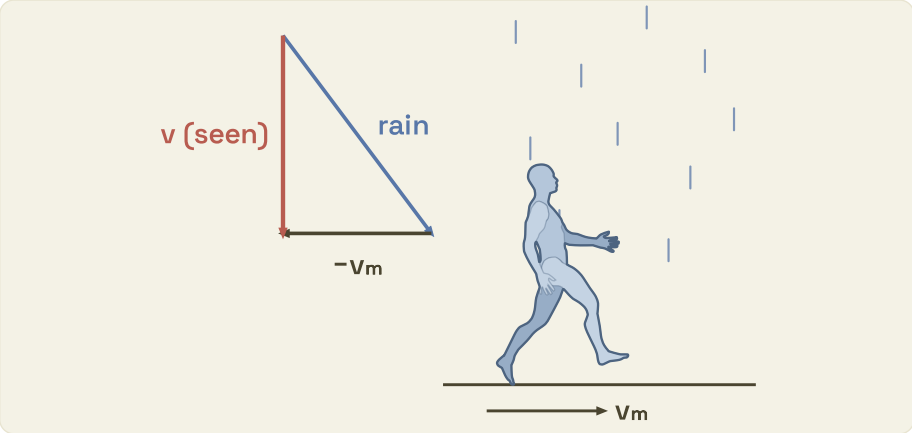


Figure 3.13 *Rain falling slantwise (its actual velocity) seen by a man walking at v_m in the direction it drifts. Adding $-v_m$ cancels the rain's sideways part, so relative to him the rain falls vertically.*

WORKED EXAMPLE

Rain on a walker

Rain falls vertically at 4 m/s. A girl walks east at 3 m/s. Find the velocity of the rain relative to her, and how she should hold her umbrella.

Answer

$v = \sqrt{4^2 + 3^2} = \mathbf{5\ m/s}$, coming from the east side, with $\tan\theta = \dfrac{3}{4}$, $\theta = 36.9^\circ$ from the vertical. She should tilt her umbrella **forward (east) by about 37°**.



Figure 3.16 *The live TV picture of David Scott's drop in 1971: the hammer and the falcon feather, falling together. They landed at the same time.*
Photo: NASA Johnson Space Center / N/A, Public domain

3.6 Projectile motion

Any body thrown into the air and then moving under **gravity alone** is a **projectile**, and its path is its **trajectory**. A stone thrown into the air, a cricket ball, a basketball shot, a jet of water from a hose, a shell fired from a gun and a parcel dropped from a flying plane are all projectiles (as long as air resistance is small enough to ignore).

The key to projectiles is that the motion splits into **two independent straight-line motions**, at right angles:

- **horizontally**, no force acts (ignoring air), so the velocity is **uniform**: $a_x = 0$;
- **vertically**, gravity acts, so the velocity changes **uniformly**: $a_y = g$ downward.

Each obeys the equations of motion separately, sharing only the time t.

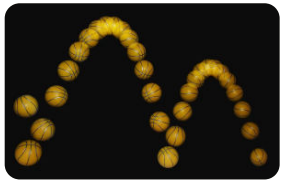


Figure 3.17 *A bouncing ball photographed by a flash at equal time intervals. Between bounces it follows parabolas: the images are nearly evenly spaced across (an almost uniform horizontal velocity; air slows it a little) but bunch together near each top, where it moves slowly up and down.*
Photo: MichaelMaggs Edit by Richard Bartz, CC BY-SA 3.0

A projectile fired horizontally

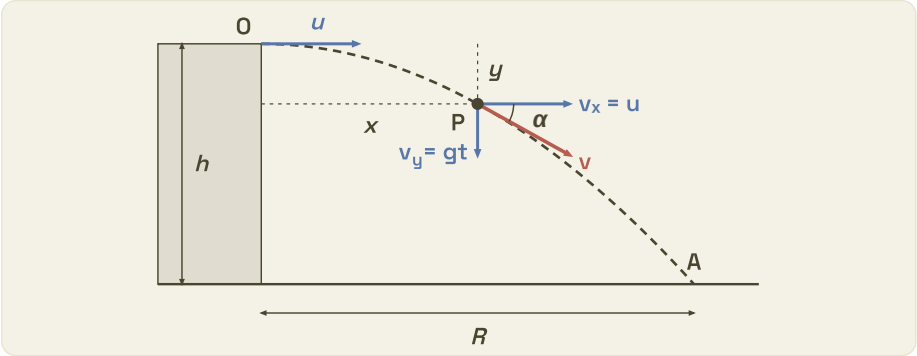


Figure 3.18 *A projectile fired horizontally at u from O, a height h above the ground. At P, after a time t, it has gone x across and y down; its velocity v has parts $v_x = u$ and $v_y = gt$, at an angle α below the horizontal. It lands at A, a range R from the foot of the launch point.*

The units look odd (m/s?) because the "1 s" of the interval is hidden: strictly, $s_n\text{th} = \left[u + \frac{a}{2}(2n-1)\right] \times 1\ \text{s}$.

A STORY FROM SCIENCE

Galileo's odd numbers

Around 1604 Galileo rolled bronze balls down a long, gently sloping groove, timing them with a water clock. He found that from rest, the distances covered in successive equal times were in the ratio 1 : 3 : 5 : 7 ..., the odd numbers. Equation (3.9) says exactly that: with $u = 0$, $s_n\text{th} = \frac{a}{2}(2n-1)$, and $2n-1$ is 1, 3, 5, 7 ... for $n = 1, 2, 3, 4$. Adding the odd numbers gives the squares (1, 4, 9, 16 ...): distance from rest grows as the square of the time, $s = \frac{1}{2}at^2$. He published it in *Two New Sciences* in 1638.

Slowing down

For a uniform **retardation** of size a, replace a by $-a$:

$$v = u - at, \qquad s = ut - \tfrac{1}{2}at^2, \qquad v^2 = u^2 - 2as, \qquad s_n\text{th} = u - \frac{a}{2}(2n-1)$$

Motion under gravity

Near the Earth's surface, without air resistance, every body falls with the same uniform acceleration, the **acceleration due to gravity**, $g \approx 9.8$ m/s² downward (taken as 10 m/s² in many problems). The equations of motion apply with $a \to g$ and $s \to h$ (height):

- **falling down** (velocity and g the same way): $v = u + gt$, $h = ut + \frac{1}{2}gt^2$, $v^2 = u^2 + 2gh$, $h_n\text{th} = u + \frac{g}{2}(2n-1)$;
- **going up** (g against the velocity): $v = u - gt$, $h = ut - \frac{1}{2}gt^2$, $v^2 = u^2 - 2gh$, $h_n\text{th} = u - \frac{g}{2}(2n-1)$.

Or, more safely: choose one direction as positive (up, say) and keep to it, so g is -9.8 m/s² all the time, going up or coming down.

WORKED EXAMPLE

A water balloon thrown down

A student throws a water balloon straight down from the top of a building at 6.0 m/s. Ignoring air resistance, and taking $g = 10$ m/s², find a. its speed after falling for 2.00 s, b. how far it falls in 2.00 s, c. its speed after falling 10.0 m.

Answer

Taking down as positive, $u = 6.0$ m/s, $a = g = 10$ m/s².

a. $v = u + gt = 6.0 + 10 \times 2.00 = \mathbf{26\ m/s}$.

b. $h = ut + \frac{1}{2}gt^2 = 6.0 \times 2.00 + \frac{1}{2} \times 10 \times 2.00^2 = 12 + 20 = \mathbf{32\ m}$.

c. $v^2 = u^2 + 2gh = 6.0^2 + 2 \times 10 \times 10.0 = 36 + 200 = 236$, so $v = \mathbf{15.4\ m/s}$.

WORKED EXAMPLE

The 12th second

A body moving in a straight line with uniform acceleration has a velocity of 25 m/s at $t = 5$ s and 34 m/s at $t = 8$ s. How far does it travel in the 12th second?

Answer

Acceleration: $a = \frac{v_2 - v_1}{t_2 - t_1} = \frac{34 - 25}{8 - 5} = 3$ m/s².

Velocity at $t = 0$: from $v = u + at$ at 5 s, $25 = u + 3 \times 5$, so $u = 10$ m/s. (It was already moving when the clock started.)

12th second:

$$s_{12}\text{th} = u + \frac{a}{2}(2n-1) = 10 + \frac{3}{2}(2 \times 12 - 1) = 10 + 34.5 = 44.5\ \text{m}$$

Check: the velocity is $10 + 3 \times 11 = 43$ m/s at 11 s and 46 m/s at 12 s, an average of 44.5 m/s over that second.

DID YOU KNOW?

The hammer and the feather

On 2 August 1971, the astronaut David Scott stood on the Moon and dropped a geologist's hammer and a falcon feather together, in front of a TV camera. With no air to slow the feather, they hit the dust at the same instant. On Earth the feather lags only because of air resistance: in a vacuum, all bodies fall with the same acceleration.

WORKED EXAMPLE

Two ships: relative velocity and closest approach

Two ships A and B are 4 km apart, A due west of B. A sails due east at 8 km/h and B due south at 6 km/h. Find a. the speed of A relative to B, and b. the shortest distance between them.

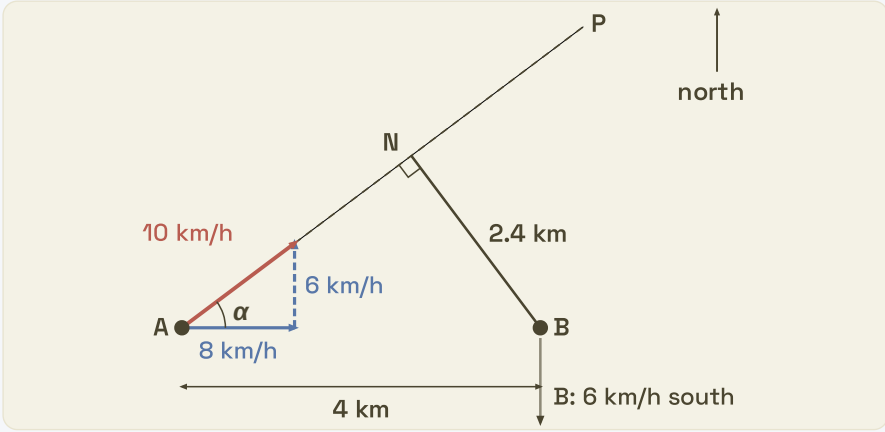


Figure 3.14 *B is held still (in B's view). A then moves with A's velocity minus B's: 8 km/h east plus 6 km/h north, 10 km/h along AP. The closest approach is the perpendicular BN from B to that path.*

Answer

a. Bring B to rest: give both ships B's velocity reversed, 6 km/h **north**. Then B is still and A moves with 8 km/h east and 6 km/h north at right angles:

$$|\vec{v}_A - \vec{v}_B| = \sqrt{8^2 + 6^2} = \sqrt{100} = 10\ \text{km/h}$$

at an angle α north of east with $\tan\alpha = \frac{6}{8}$, so $\sin\alpha = 0.6$ and $\cos\alpha = 0.8$.

b. In B's view, A moves along the straight line AP from its start, 4 km west of B. The closest A comes to B is the perpendicular distance BN from B to that line:

$$BN = AB\sin\alpha = 4 \times 0.6 = 2.4\ \text{km}$$

A reaches N after traveling $AN = AB\cos\alpha = 4 \times 0.8 = 3.2$ km at 10 km/h, that is after **0.32 h** (about 19 minutes). The shortest distance is **2.4 km**.

DID YOU KNOW?

Air-to-air refueling

A tanker airplane can pass fuel to a fighter through a hose while both fly at hundreds of kilometers per hour. The trick is relative velocity: matched to within a few kilometers per hour, the two planes are almost at rest relative to each other, and the hose can stay connected.

3.5 Equations of motion

For a body moving in a straight line with **uniform acceleration**, five quantities describe the motion:

Symbol	Quantity
s	displacement (the distance traveled, if it doesn't turn back)
u	initial velocity
v	final velocity
a	acceleration (constant)
t	time taken

Their initials spell **suvat**, an easy way to remember them. Any three give the other two, through three equations, derived below in three ways.

From the definitions

First equation. By the definition of acceleration, $a = \dfrac{v-u}{t}$, so

$$v = u + at \tag{3.1}$$

Second equation. Under uniform acceleration the velocity grows steadily, so the average velocity is halfway between the first and last: $\dfrac{u+v}{2}$. The displacement is the average velocity × time:

$$s = \frac{(u+v)}{2}\,t$$

From (3.1), $t = \dfrac{v-u}{a}$. Substituting,

$$s = \frac{(u+v)}{2} \times \frac{(v-u)}{a} = \frac{v^2-u^2}{2a} \quad\Rightarrow\quad v^2 = u^2 + 2as \tag{3.2}$$

Third equation. Put $v = u + at$ into $s = \dfrac{(u+v)}{2}t$:

$$s = \frac{(u+u+at)}{2}\,t \quad\Rightarrow\quad s = ut + \tfrac{1}{2}at^2 \tag{3.3}$$

From the velocity–time graph

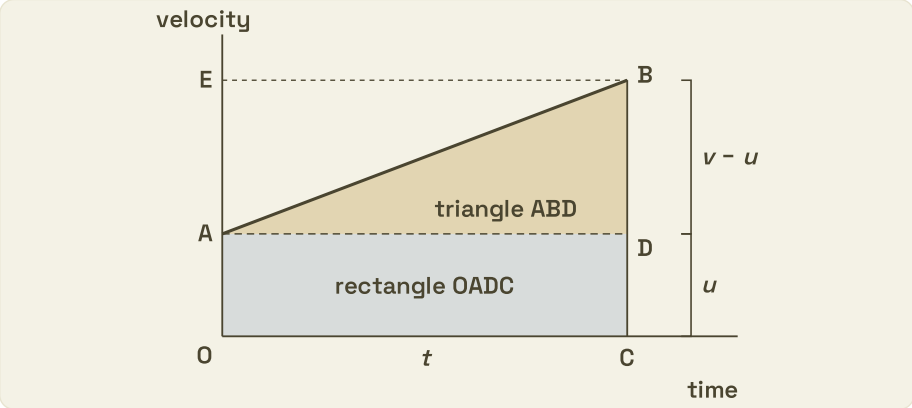


Figure 3.15 *Uniform acceleration from u to v in time t. OA = CD = u; BC = v; OC = AD = t; BD = v − u. The slope of AB is the acceleration; the area of the trapezium OABC is the displacement.*

A body moves with uniform acceleration: its velocity is u at time 0 and v at time t. Its velocity–time graph is the straight line AB (Figure 3.15), with OA = CD = u, BC = v, OC = AD = t and BD = BC − CD = $v - u$.

First equation, from the slope. The slope of AB is the acceleration:

$$a = \frac{BD}{AD} = \frac{v-u}{t} \quad\Rightarrow\quad v = u + at \tag{3.4}$$

Second equation, from the area as a trapezium. The displacement is the area of the trapezium OABC, with parallel sides OA and BC and width OC:

$$s = \tfrac{1}{2}(OA + BC) \times OC = \tfrac{1}{2}(u+v)\,t$$

From the slope, $OC = t = \dfrac{v-u}{a}$, so

$$s = \tfrac{1}{2}(u+v) \times \frac{(v-u)}{a} = \frac{v^2-u^2}{2a} \quad\Rightarrow\quad v^2 = u^2 + 2as \tag{3.5}$$

Third equation, from the area in two pieces. Split the trapezium into the rectangle OADC and the triangle ABD:

$$s = OA \times OC + \tfrac{1}{2} \times AD \times BD = ut + \tfrac{1}{2} \times t \times (v-u)$$

and since $v - u = at$,

$$s = ut + \tfrac{1}{2}at^2 \tag{3.6}$$

By calculus

The graph proof works only for a straight line. Calculus gives the same equations straight from the definitions $a = \dfrac{dv}{dt}$ and $v = \dfrac{ds}{dt}$, for constant a:

- $\dfrac{dv}{dt} = a$. Integrate from $t = 0$ (velocity u) to t (velocity v): $\displaystyle\int_u^v dv = \int_0^t a\,dt$, so $v - u = at$, that is $v = u + at$.
- $\dfrac{ds}{dt} = v = u + at$. Integrate from position 0 to s: $\displaystyle\int_0^s ds = \int_0^t (u + at)\,dt$, so $s = ut + \tfrac{1}{2}at^2$.
- By the chain rule, $a = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = v\dfrac{dv}{ds}$. So $v\,dv = a\,ds$; integrating, $\tfrac{1}{2}(v^2 - u^2) = as$, that is $v^2 = u^2 + 2as$.

CAREFUL

Only for uniform acceleration

The three equations hold only when the acceleration is **constant**. For a car whose acceleration changes (like the next example), go back to $v = \dfrac{dx}{dt}$ and $a = \dfrac{dv}{dt}$; using $v = u + at$ there gives wrong answers.

WORKED EXAMPLE

A car pulls away from a traffic light

A car is stopped at a traffic light. It then travels along a straight road so that its distance from the light is $x = bt^2 - ct^3$, where $b = 2.40$ m/s² and $c = 0.120$ m/s³.

a. Find its average velocity from $t = 0$ to $t = 10.0$ s.

b. Find its instantaneous velocity at (i) $t = 0$, (ii) $t = 5.0$ s, (iii) $t = 10.0$ s.

c. How long after starting from rest is the car again at rest?

Answer

a. $x(0) = 0$; $x(10.0) = 2.40 \times 10.0^2 - 0.120 \times 10.0^3 = 240 - 120 = 120$ m.

$$v_{\text{av}} = \frac{120 - 0}{10.0 - 0} = 12.0 \text{ m/s}$$

b. $v = \dfrac{dx}{dt} = 2bt - 3ct^2$.

i. At $t = 0$: $v = 0$.

ii. At $t = 5.0$ s: $v = 2 \times 2.40 \times 5.0 - 3 \times 0.120 \times 5.0^2 = 24.0 - 9.0 = \mathbf{15.0\ m/s}$.

iii. At $t = 10.0$ s: $v = 48.0 - 36.0 = \mathbf{12.0\ m/s}$.

c. $v = 0$ when $2bt - 3ct^2 = t(2b - 3ct) = 0$: at $t = 0$ (the start) or

$$t = \frac{2b}{3c} = \frac{2 \times 2.40}{3 \times 0.120} = 13.3 \text{ s}$$

Its acceleration $a = 2b - 6ct$ changes with time, which is why the equations of motion can't be used here.

Check 8

A body starts from rest with a uniform acceleration of 2 m/s². How far does it go in 5 s?

a 10 m b 25 m
c 50 m d 5 m

Distance traveled in the nth second

The **nth second** is the one-second interval from $t = n - 1$ to $t = n$. For uniform acceleration, the distance in the first n seconds and in the first $n - 1$ seconds are

$$s_n = un + \tfrac{1}{2}an^2, \qquad s_{n-1} = u(n-1) + \tfrac{1}{2}a(n-1)^2$$

Subtracting,

$$s_n\text{th} = un + \tfrac{1}{2}an^2 - un + u - \tfrac{1}{2}an^2 + an - \tfrac{1}{2}a = u + an - \tfrac{1}{2}a$$

$$s_n\text{th} = u + \frac{a}{2}(2n - 1) \tag{3.9}$$