Unit 3

Answers: Kinematics

Quick checks

1 a You and the driver move together, so the distance and direction between you don't change: relative to the driver you are at rest. Relative to the road you move north at 15 m/s, and relative to the road the road-side trees seem to rush south at 15 m/s.

2 c After one lap she is back at the start; the extra half lap takes her to the opposite side of the circle, one diameter away: $2 \times 50 = 100$ m. The distance she ran is $1.5 \times 2\pi \times 50 \approx 471$ m. Zero would be after a whole number of laps.

3 c A raindrop soon reaches a steady speed (its weight balanced by air resistance) and falls straight down: same speed, same direction. The fan tip and the car keep changing direction; the stone keeps speeding up.

4 b Horizontal means the position doesn't change: the body is at rest. A steady speed is a sloping straight line; a backwards motion slopes down; an acceleration makes the line curve.

5 b The velocity is east and getting smaller, so the change in velocity points west, and the acceleration points the way the velocity changes: west. Moving east with a westward acceleration is exactly what braking is.

6 b Slope $= (0 - 20) \div 4 = -5$ m/s², a deceleration. Area $= \frac{1}{2} \times 4 \times 20 = 40$ m. 80 m forgets the half in the triangle's area.

7 c Take north as positive: $v_B - v_A = (-15) - (+20) = -35$ m/s, that is 35 m/s south. To a passenger in A, B rushes south at the two speeds added.

8 b $s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2} \times 2 \times 5^2 = 25$ m. 50 m forgets the half; 10 m is its velocity after 5 s (in m/s), not a distance.

9 b $T = \sqrt{2h/g} = \sqrt{2 \times 20/10} = 2$ s, and $R = uT = 15 \times 2 = 30$ m. The fall takes as long as if it had been dropped; the 15 m/s only carries it across.

10 c At the top the vertical part is zero, but the horizontal part never changes: $u \cos 30^\circ = 20 \times 0.866 = 17.3$ m/s. Zero is the vertical part only; 10 m/s is $u \sin 30^\circ$, the vertical part at launch.

11 c Let each third be d. Total time $= \dfrac{d}{10} + \dfrac{d}{20} + \dfrac{d}{60} = \dfrac{6d + 3d + d}{60} = \dfrac{d}{6}$. Average speed $= \dfrac{3d}{d/6} = 18$ km/h. The plain average of the three speeds, 30 km/h, is wrong because the bus spends much longer at 10 km/h.

12 b $a = \dfrac{20 - 10}{5} = 2$ m/s². Three seconds earlier, $v = 10 - 2 \times 3 = 4$ m/s.

13 **c** $S_1 = \frac{1}{2}a(10)^2 = 50a$; in 20 s it goes $\frac{1}{2}a(20)^2 = 200a$, so $S_2 = 200a - 50a = 150a = 3S_1$. (Galileo's odd numbers again: 1 : 3.)

14 **b** $s = kt$ gives $v = \dfrac{ds}{dt} = k$, a constant, so $a = \dfrac{dv}{dt} = 0$: uniform velocity.

15 **a** In an hour the tip goes once round and returns to where it started: zero displacement, so zero average velocity. But it has traveled a whole circumference, so the distance and the average speed are not zero.

16 **a** In the first 40 cm, $v^2 = u^2 - 2as$ with $v = u/2$: $\dfrac{u^2}{4} = u^2 - 2a(40)$, so $2a = \dfrac{3u^2}{4 \times 40}$.

Then from $u/2$ to rest: $0 = \dfrac{u^2}{4} - 2a\,x$, so $x = \dfrac{u^2/4}{2a} = \dfrac{u^2}{4} \times \dfrac{160}{3u^2} = \dfrac{40}{3}$ cm $\approx$ 13.3 cm.

Shortcut: the stopping distance goes as v^2. Losing the first half of its speed uses up ¾ of its v^2; the last half uses the last ¼, a third as much: 40 ÷ 3 cm.

17 **b** After half a lap she is a diameter away from the start: displacement = 2 × 25 = 50 m. Average velocity = 50 ÷ 25 = 2 m/s. π m/s is her average **speed** (half the circumference, 25π m, in 25 s).

18 **c** 54 km/h = 15 m/s and 90 km/h = 25 m/s (divide by 3.6). They move in opposite directions, so the relative velocity is 15 + 25 = 40 m/s (southward, as seen from B).

19 **b** *Rectilinear* means "in a straight line". Plane, curvilinear and circular motions all bend.

20 **b** At a steady speed, the speed at every instant equals the average: 90 miles ÷ 3 h = 30 miles per hour (mph). Without that, 30 mph is only the average, and the speed at 2 p.m. could be anything.

21 **d** With no air resistance, every body falls with the same acceleration g, whatever its mass or shape: the hammer and the feather on the Moon.

22 **b** Let the fall take T. In T it falls $\frac{1}{2}gT^2$; in the first $T - 1$ seconds, $\frac{1}{2}g(T-1)^2$. The last second covers half the height, so the first $T - 1$ seconds cover the other half:

$$\tfrac{1}{2}g(T-1)^2 = \tfrac{1}{2} \times \tfrac{1}{2}gT^2 \Rightarrow (T-1)^2 = \frac{T^2}{2} \Rightarrow T - 1 = \frac{T}{\sqrt{2}} \Rightarrow T = \frac{\sqrt{2}}{\sqrt{2}-1} = 2 + \sqrt{2} \approx 3.41 \text{ s}.$$

(The other root, $2 - \sqrt{2}$ s, is less than 1 s and has no "last second". The answer doesn't depend on g.)

23 **d** The third stage is the triangle from 25 s to 60 s, falling from 12 m/s to 0: area = ½ × (60 − 25) × 12 = ½ × 35 × 12 = 210 m.

24 **b** The graph is flat (BC) from 10 s to 30 s: the position doesn't change, so the car is at rest for 30 − 10 = 20 s.

25 b Area above the axis (0 to 6 s): triangle ½ × 6 × 4 = +12 m.
Area below the axis: from 6 to 8 s, triangle ½ × 2 × 4 = 4 m; from 8 to 12 s, rectangle 4 × 4 = 16 m; from 12 to 16 s, triangle ½ × 4 × 4 = 8 m; in all −28 m.
Displacement = 12 − 28 = −16 m. (The distance traveled is 12 + 28 = 40 m.)

26 c The area is a trapezium with parallel sides 100 s (the base) and 60 − 20 = 40 s (the top), and height v: $\frac{1}{2}(100 + 40)v = 70v = 1750$, so $v = 25$ m/s.

27 b In the last 5 s (20 s to 25 s) the speed falls from 16 m/s to 0: deceleration = 16 ÷ 5 = 3.2 m/s².

28 c The velocity is a steady 20 m/s, so the area is a rectangle: 20 × 20 = 400 m.

19.

a. **Average speed** is the total distance traveled divided by the total time taken (a scalar). **Average velocity** is the total displacement divided by the total time taken (a vector, in the direction of the displacement). They are equal in size only for motion in a straight line without turning back.

b. $v = \dfrac{dx}{dt} = 0 - 3Bt^2 + 4At^3 = 4At^3 - 3Bt^2.$

$a = \dfrac{dv}{dt} = 12At^2 - 6Bt.$

c. It is impossible for two reasons:

- a vertical line through the loop cuts it twice: the body would be at **two positions at the same instant**;
- along the lower half of the loop the line runs to the left, so **time would be running backwards**.

A position–time graph must give one position at each time and always move forward in time.

20.

a. Yes. Displacement is measured from a chosen origin, with one direction positive. A body on the other side of the origin has a **negative displacement**, and its graph lies below the time axis (for example, a ball that falls past the point it was thrown from, taking up as positive).

b. i. Distance = 4 × 1609 = 6436 m; time = 7 × 60 = 420 s. Average speed = 6436 ÷ 420 = **15.3 m/s**.

ii. After 4 complete laps he is back at the start: displacement = 0, so his average velocity = **0**.

c. Average speed uses the **distance**, the whole length of track he ran (6436 m). Average velocity uses the **displacement**, the change in his position, which is zero because he ended where he started.

21.

a. A **projectile** is a body thrown into the air (given an initial velocity) that then moves under the action of gravity alone. Its path is its trajectory.

b. Fired horizontally at u from O, with x across and y downward. Horizontally there is no acceleration: $x = ut$, so $t = x/u$. Vertically it starts at 0 with acceleration g: $y = \frac{1}{2}gt^2$. Substituting, $y = \frac{1}{2}g\left(\frac{x}{u}\right)^2 = \frac{g}{2u^2}x^2$. Since $\frac{g}{2u^2}$ is a constant, this is $y = kx^2$, a parabola.

c. $R_{30} = \dfrac{u^2 \sin 60^\circ}{g} = 2$ km, so $\dfrac{u^2}{g} = \dfrac{2}{\sin 60^\circ} = \dfrac{2}{\sqrt{3}/2} = \dfrac{4}{\sqrt{3}}$ km.

$R_{45} = \dfrac{u^2 \sin 90^\circ}{g} = \dfrac{u^2}{g} = \dfrac{4}{\sqrt{3}}$ km $\approx$ **2.31 km**.

22.

a. The 45° rule assumes a launch from ground level, no air, and the same speed at every angle. None of these holds for a javelin: it is released about 2 m above the ground (which favors a flatter angle); it is aerodynamic, so air gives it some lift and drag; and a thrower's body can give it more speed at a flatter angle than at a steep one. Together these make the best release angle about 32–36°.

b. $R_\theta = \dfrac{u^2 \sin 2\theta}{g}$, and $R_{90^\circ-\theta} = \dfrac{u^2 \sin(180^\circ - 2\theta)}{g} = \dfrac{u^2 \sin 2\theta}{g}$, since $\sin(180^\circ - x) = \sin x$. They are equal.

With a given launcher, which fires at a fixed speed, the range is changed mainly by the **angle of projection** (and, at a different place, by g); if the launcher's speed can be set, the speed changes it most, since R goes as u^2.

c. Project it at **45°**: then $\sin 2\theta = 1$ and $R = \dfrac{u^2}{g}$ is as large as it can be for a given speed, so a given range needs the least speed, $u = \sqrt{Rg}$.

23.

a. When released, the package has the plane's horizontal velocity, and it keeps moving forward at that velocity while it falls. Released directly above the target, it would land beyond it. So it is released earlier, a horizontal distance $uT = u\sqrt{2h/g}$ before the target.

b. They hit the ground **together**. Both start with zero vertical velocity and have the same vertical acceleration g, so both take $T = \sqrt{2h/g}$; the horizontal motion doesn't affect the vertical one.

c. Find its two parts when it lands: horizontally, $v_x = u$ (unchanged); vertically, $v_y = gT = \sqrt{2gh}$ (from $v_y^2 = 2gh$). Combine them: $v = \sqrt{u^2 + 2gh}$, at an angle α below the horizontal with $\tan\alpha = \dfrac{\sqrt{2gh}}{u}$.

24. a. A **projectile** is a body given an initial velocity and then moving under gravity alone.
Fire it at u at θ: $u_x = u\cos\theta$, $u_y = u\sin\theta$; horizontally no acceleration, vertically $-g$ (up positive).

Time of flight: it lands when $y = 0$: $0 = u\sin\theta\, T - \frac{1}{2}gT^2$, so (apart from $T = 0$) $T = \dfrac{2u\sin\theta}{g}$.
Maximum height: at the top $v_y = 0$; from $v_y^2 = u_y^2 - 2gH$, $0 = u^2\sin^2\theta - 2gH$, so $H = \dfrac{u^2\sin^2\theta}{2g}$.
Range: $R = u_x T = u\cos\theta \times \dfrac{2u\sin\theta}{g} = \dfrac{u^2\sin 2\theta}{g}$.

b. The range is greatest at $\theta = 45°$: $R_{\max} = \dfrac{u^2}{g}$. The height of that flight is $H = \dfrac{u^2\sin^2 45^\circ}{2g} = \dfrac{u^2}{4g}$. So $R_{\max} = 4H$.

c. Complementary angles give the same range: $90° - 18° = \mathbf{72°}$.

25. a. Fired at u at θ, with x across and y up: $x = u\cos\theta\, t$, so $t = \dfrac{x}{u\cos\theta}$; and $y = u\sin\theta\, t - \frac{1}{2}gt^2$. Substituting,
$y = x\tan\theta - \dfrac{g}{2u^2\cos^2\theta}x^2$.
For a given launch, $\tan\theta$ and $\dfrac{g}{2u^2\cos^2\theta}$ are constants, so $y = bx - cx^2$: a parabola.

b. Its vertical motion starts from zero vertical velocity with acceleration g: $h = \frac{1}{2}gT^2$, so $T = \sqrt{\dfrac{2h}{g}}$, the same as a dropped body.

c. At the **highest point**. The acceleration is always g, straight down; at the top the vertical velocity is zero, so the velocity is horizontal, at 90° to g. (For a projectile fired horizontally, that is the launch point.)

26.

a. From 0 to 2 s the velocity rises from 0 to 8 m/s: $a = \dfrac{8-0}{2} = \mathbf{4\ m/s^2}$. From $F = ma$ (Unit 4), $m = \dfrac{F}{a} = \dfrac{10}{4} = \mathbf{2.5\ kg}$.

b. From 10 s to 12 s it falls from 8 m/s to 4 m/s: $a = \dfrac{4-8}{2} = \mathbf{-2\ m/s^2}$ (a deceleration of 2 m/s²).

c. The area under the graph, piece by piece:

- 0 to 2 s, triangle: ½ × 2 × 8 = 8 m;
- 2 to 10 s, rectangle: 8 × 8 = 64 m;
- 10 to 12 s, trapezium: ½ × (8 + 4) × 2 = 12 m;
- 12 to 16 s, rectangle: 4 × 4 = 16 m.

Total displacement = 8 + 64 + 12 + 16 = **100 m**.

27.

a. A trapezium: a straight line rising from (0, 0) to (8 s, 10 m/s); a flat line at 10 m/s from 8 s to (8 + T) s; a straight line falling to 0 at 100 s.

b. The area is the distance, 800 m:

- speeding up: ½ × 8 × 10 = 40 m;
- steady: 10T m;
- slowing down, over the remaining $100 - 8 - T = 92 - T$ s: ½ × (92 − T) × 10 = 460 − 5T m.

$40 + 10T + 460 - 5T = 800 \Rightarrow 500 + 5T = 800 \Rightarrow T = \mathbf{60\ s}$.

28.

a. From 0 to 4 s it goes from 0 to 3 m/s: $a = \dfrac{3}{4} = \mathbf{0.75\ m/s^2}$.

b. From 7 s to 10 s it goes from 6 m/s to 0: $a = \dfrac{0-6}{10-7} = -2$ m/s², a **deceleration of 2.0 m/s²**.

c. Area from 0 to 7 s:

- 0 to 4 s, triangle: ½ × 4 × 3 = 6 m;
- 4 to 6 s, rectangle: 2 × 3 = 6 m;
- 6 to 7 s, trapezium: ½ × (3 + 6) × 1 = 4.5 m.

Total = **16.5 m**.

29. i. $t = \dfrac{v-u}{a} = \dfrac{15-10}{1} = \mathbf{5\ s}$.

ii. $s = \dfrac{v^2-u^2}{2a} = \dfrac{225-100}{2} = \mathbf{62.5\ m}$.

iii. $v^2 = u^2 + 2as = 10^2 + 2\times 1\times 100 = 300$, so $v = \mathbf{17.3\ m/s}$. (This assumes it goes on accelerating past 15 m/s, which it reaches at 62.5 m; if it stopped accelerating there, it would still be at 15 m/s at 100 m.)

30. i. Up as positive: it returns when its displacement is 0: $0 = 20t - \frac{1}{2}\times 10\times t^2 = t(20-5t)$, so $t = \mathbf{4\ s}$.

ii. At the top $v = 0$: $0 = 20^2 - 2\times 10\times H$, so $H = \dfrac{400}{20} = \mathbf{20\ m}$.

31. Speed on hitting the ground: $v = \sqrt{2gh} = \sqrt{2\times 10\times 20} = 20$ m/s.
Rebound speed: $\frac{3}{4}\times 20 = 15$ m/s, upward.
It goes up and comes back down to the ground: time $= \dfrac{2u}{g} = \dfrac{2\times 15}{10} = \mathbf{3\ s}$.

32. i. $t = \dfrac{u\sin\theta}{g} = \dfrac{320\times 0.5}{10} = \mathbf{16\ s}$.

ii. $R = \dfrac{u^2\sin 2\theta}{g} = \dfrac{320^2\times \sin 60^\circ}{10} = \dfrac{102\,400\times 0.866}{10} = \mathbf{8868\ m}$.

iii. At 45°: $R_{\max} = \dfrac{u^2}{g} = \dfrac{102\,400}{10} = \mathbf{10\ 240\ m}$.

33. Measure heights up from the ground. After t seconds:
dropped ball: $y_1 = 60 - \frac{1}{2}\times 10t^2 = 60 - 5t^2$;
thrown ball: $y_2 = 20t - 5t^2$.
They meet when $y_1 = y_2$: $60 - 5t^2 = 20t - 5t^2 \Rightarrow 20t = 60 \Rightarrow t = \mathbf{3\ s}$.
Height: $y_1 = 60 - 5\times 9 = \mathbf{15\ m\ above\ the\ ground}$.
(Both have the same acceleration, so relative to each other they move at a steady 20 m/s and close 60 m in 3 s. The thrown ball is already coming down by then: it reached its top, 20 m, at 2 s.)

34. $a = \dfrac{34-25}{8-5} = 3$ m/s². From $25 = u + 3\times 5$, $u = 10$ m/s.
$s_{12} = u + \dfrac{a}{2}(2n-1) = 10 + \dfrac{3}{2}\times 23 = 10 + 34.5 = \mathbf{44.5\ m}$.

35. On Earth: $R_E = \dfrac{u^2\sin 90^\circ}{g} = \dfrac{100}{10} = \mathbf{10\ m}$.
On the Moon, $g_M = \dfrac{10}{6}$ m/s²: $R_M = \dfrac{100}{10/6} = \mathbf{60\ m}$, six times as far. The range is inversely proportional to g.

36. $R = \dfrac{u^2 \sin 2\theta}{g} = \dfrac{900 \times \sin 60^\circ}{10} = \dfrac{900 \times 0.866}{10} = \mathbf{77.9\ m}$.
It lands 77.9 m away, beyond the 75 m boundary, without touching the ground first: **a six** (in this ideal case; real air resistance would cut the range, and fielders would find out the rest).

37. a. $R = \dfrac{u^2 \sin 2\theta}{g}$, so $u = \sqrt{\dfrac{Rg}{\sin 54^\circ}} = \sqrt{\dfrac{7.80 \times 9.8}{0.809}} = \sqrt{94.5} = \mathbf{9.72\ m/s}$.

b. $R \propto u^2$, so the new range is $7.80 \times 1.05^2 = 7.80 \times 1.1025 = \mathbf{8.60\ m}$, 0.80 m longer.

38. $R = \dfrac{u^2 \sin 2\theta}{g} = \dfrac{121 \times \sin 40^\circ}{9.8} = \dfrac{121 \times 0.643}{9.8} = \mathbf{7.94\ m}$.
$H = \dfrac{u^2 \sin^2 \theta}{2g} = \dfrac{121 \times 0.342^2}{19.6} = \mathbf{0.72\ m}$.
(This treats the jumper as a point launched from and landing at the same level; a real jumper's center of mass starts higher than it lands, which adds a little.)

39. a. $T = \dfrac{2u \sin \theta}{g} = \dfrac{2 \times 27 \times 0.5}{9.8} = \mathbf{2.76\ s}$ (about 2.8 s).

b. $H = \dfrac{u^2 \sin^2 \theta}{2g} = \dfrac{729 \times 0.25}{19.6} = \mathbf{9.3\ m}$.

c. $R = \dfrac{u^2 \sin 2\theta}{g} = \dfrac{729 \times 0.866}{9.8} = \mathbf{64\ m}$.

40. $u_x = 7.0 \cos 53.0^\circ = 4.21$ m/s (unchanged all the way); $u_y = 7.0 \sin 53.0^\circ = 5.59$ m/s upward.
Vertically, up positive, its displacement on landing is -50.0 m:
$v_y^2 = u_y^2 + 2(-9.8)(-50.0) = 31.2 + 980 = 1011$, so $v_y = 31.8$ m/s, downward.
$v = \sqrt{4.21^2 + 31.8^2} = \sqrt{17.7 + 1011} = \mathbf{32\ m/s}$.
Direction: $\tan \alpha = \dfrac{31.8}{4.21} = 7.55$, so $\alpha = \mathbf{82°}$ below the horizontal.
(Energy gives the speed more quickly, Unit 5: $v = \sqrt{u^2 + 2gh} = \sqrt{49 + 980} = 32$ m/s, whatever the angle.)