30. Three vectors satisfy $\vec{C} = \vec{A} + \vec{B}$, with magnitudes 3, 4 and 5 units. Find the angle between $\vec{A}$ and $\vec{C}$.

31. A man who has lost his way drives 3.25 km north, then 4.75 km west, and finally 1.5 km south. Using components, find the magnitude and direction of his resultant displacement.

32. $\vec{A} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{B} = 3\hat{i} - 4\hat{j} + 5\hat{k}$. Find the area of the parallelogram they form.

33. A particle has a displacement of 8 m. One of its rectangular components is 4 m. What is the other?

34. A rocket fires two engines at once. One gives a thrust of 725 N straight forward, the other 513 N at 32.4° above the forward direction. Find the magnitude and direction (relative to forward) of the resultant force.

35. $\vec{A} = -2.00\hat{i} + 3.00\hat{j} + 4.00\hat{k}$ and $\vec{B} = 3.00\hat{i} + 1.00\hat{j} - 3.00\hat{k}$.
    a. Write $\vec{A} - \vec{B}$ using unit vectors.
    b. Find the magnitude of each vector.
    c. Find the magnitude of $\vec{A} - \vec{B}$. Is it the same as the magnitude of $\vec{B} - \vec{A}$? Explain.

36. A caver is surveying a cave. She follows a passage 180 m straight west, then 210 m in a direction 45° east of south, then 280 m at 30° east of north. After a fourth, unmeasured displacement she is back where she started. Find the magnitude and direction of the fourth displacement.

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

Unit 2

Vectors

This unit online, with its interactives:
science-playground.com/s/11f2

Pokhara is about 142 km from Kathmandu in a straight line. A pilot who takes off from Tribhuvan Airport and flies 142 km will not land in Pokhara unless she flies the right way: about 23° north of west. Fly 142 km due south and you are over India. The number alone is not enough; the direction is half the information.

Quantities like this, which need a direction as well as a size, are **vectors**. Forces, velocities, accelerations and displacements are all vectors, and most of physics is written in them. This unit teaches the language: how to draw vectors, add them, take them apart into pieces, and multiply them in the two ways nature uses.

A STORY FROM SCIENCE

Two men who made physics shorter

In 1865 James Clerk Maxwell published his theory of electricity and magnetism. It was written out component by component, in twenty equations with twenty unknowns, and very few people could follow it. Two men, working separately, saw that most of the clutter could go.

Josiah Willard Gibbs (1839–1903), born in New Haven in the United States, earned in 1863 the first doctorate in engineering ever given in America, at Yale, and stayed at Yale all his life. To teach his students, he wrote a short pamphlet on vector analysis, printed privately in 1881 and 1884.

Oliver Heaviside (1850–1925) was an English telegraph clerk who left school at sixteen and taught himself mathematics. Working alone, he rewrote Maxwell's equations in vectors: the twenty became the four that every physics student learns today.

Both took the idea from the Irish mathematician William Rowan Hamilton, who had coined the word *vector* (from the Latin *vehere*, to carry) in the 1840s, and both threw away the parts of Hamilton's system they didn't need. Hamilton's followers were furious, and the argument went on in journals for years. The vectors of this unit, with their dot and cross products, are Gibbs's and Heaviside's.

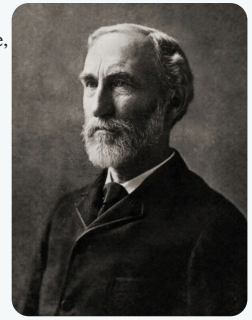


Figure 2.1 *Josiah Willard Gibbs (1839–1903), one of the two inventors of vector analysis.*
Photo: Unknown. Uploaded by Serge Lachinov (обработка для wiki), Public domain

2.1 Introduction

Think about these:

1. Pokhara lies about 142 km from Kathmandu as the crow flies (about 200 km by road). Can a pilot land in Pokhara by flying 142 km in **any** direction from Kathmandu?
2. A screw is turned clockwise. Which way does it move? And if it is turned anticlockwise?
3. Your mother asks you to fetch something from a shop and tells you only "it's 200 m from the main gate". Can you find the shop?

In each case the size alone is not enough: you need a **direction** too. The pilot needs a heading; the screw goes in or comes out depending on which way it turns; and 200 m from the gate could be anywhere on a circle round it.

Physical quantities are of two kinds:

- A **scalar** has only a **magnitude** (a size: a number with its unit). Mass, length, time, temperature, distance, speed, work, energy, and the money in your pocket are scalars. A hundred-rupee note is worth a hundred rupees whichever way it points.
- A **vector** has a **magnitude and a direction**, and combines with other vectors by the rules of vector addition (the triangle law, Section 2.3). Displacement, velocity, acceleration, force and momentum are vectors.

That second condition matters. **Electric current** has a size and a direction along the wire, yet it is a scalar: when two currents meet at a junction, the current that leaves is simply their sum, whatever the angle between the wires. Currents don't add by the triangle law, so current is not a vector.

Check 1
Which of these is a vector?

a speed
b work
c momentum
d electric current

How a vector is shown

In a drawing, a vector is an arrow. Its **length**, to a chosen scale, is the vector's magnitude, and the way it points is the vector's direction. The arrow starts at its **initial point** (tail) and ends at its **terminal point** (head).

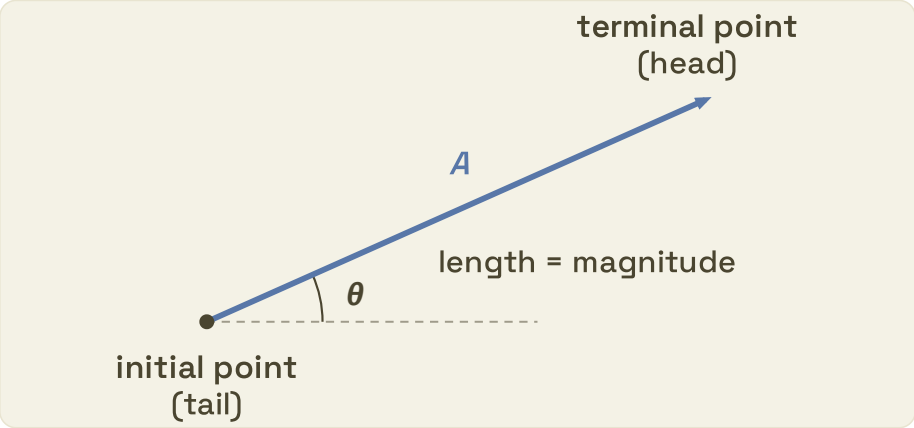


Figure 2.2 A vector drawn as an arrow. Its length (to scale) gives the magnitude, and its direction is given by an angle θ from a chosen line.

In writing, a vector is a letter with an arrow over it, $\vec{A}$, or in print a **bold** letter, **A**. Its magnitude is written $|\vec{A}|$, or simply A in plain italic. So "**F** = 20 N, east" is a vector, and F = 20 N is its magnitude.

A vector doesn't have a fixed place: moving an arrow without turning it or changing its length gives **the same vector**. A walk of 3 km north is the same displacement from Bhaktapur or from Butwal.

2.2 Types of vectors

Unit vector

A **unit vector** has a magnitude of exactly **1**; it carries only a direction. The unit vector along a vector $\vec{A}$ is written $\hat{A}$ or $\hat{n}$ ("A hat"), and is $\vec{A}$ divided by its own magnitude:

$$\hat{n} = \frac{\vec{A}}{A} \qquad \text{so} \qquad \vec{A} = A\,\hat{n}$$

that is, any vector is its magnitude times a unit vector in its direction.

Check 23
16. The angle made by $\vec{A} = 2\hat{i} + 2\hat{j}$ with the X-axis is:

a 0°
b 30°
c 45°
d 60°

Check 24
17. Three vectors satisfy $\vec{A}\cdot\vec{B} = 0$ and $\vec{A}\cdot\vec{C} = 0$. Then $\vec{A}$ is parallel to:

a $\vec{B}\cdot\vec{C}$
b $\vec{B}\times\vec{C}$
c $\vec{B}$
d $\vec{C}$

Check 25
18. The largest possible magnitude of $\vec{A} - \vec{B}$ is:

a A + B
b A − B
c $\sqrt{A^2 + B^2}$
d $\sqrt{A^2 - B^2}$

23.
a. Is a quantity with magnitude and direction necessarily a vector?
b. State and explain the polygon law of vector addition.
c. Show that the magnitude of $\vec{A}\times\vec{B}$ is the area of a parallelogram.

24.
a. How is a vector shown in symbols and in a drawing?
b. State and explain the polygon law of vector addition.
c. Two equal vectors are inclined at 120°. Prove that their resultant is equal to either vector.

25.
a. In what two ways can two vectors be multiplied?
b. State and explain the parallelogram law of vector addition. What happens to the resultant when the two vectors act at 180°?
c. What is a unit vector? Explain how to resolve a vector into two rectangular components. Why is $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$? What are the conditions for two vectors to be perpendicular (orthogonal) and parallel?

26.
a. State the triangle law of vector addition.
b. Derive expressions for the magnitude and direction of the resultant of two vectors inclined at an angle θ.
c. Two vectors have magnitudes A + B and A − B. Find the angle between them if their resultant is $\sqrt{3A^2 + B^2}$.

Numerical problems

27. Vectors $\vec{P}$ and $\vec{Q}$ have magnitudes 3 units and 4 units. What is the angle between them if their dot product is a. zero, b. 12 units, c. −12 units?

28. $\vec{A} = 4.00\hat{i} + 3.00\hat{j}$ and $\vec{B} = 5.00\hat{i} - 2.00\hat{j}$.
a. Find the magnitude of each vector.
b. Write the vector difference $\vec{A} - \vec{B}$ using unit vectors.
c. Find the magnitude and direction of $\vec{A} - \vec{B}$.

29. A man pushes on the ground with a force of 100 N at 30° to the vertical. Find the components of the ground's reaction along the vertical and the horizontal.

Check 15

8. A vector's x- and y-components are equal. Its angle with the X-axis is:

a 30° b 45°
c 60° d 0°

Check 16

9. Vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ have magnitudes 3, 4 and 12. The magnitude of $\vec{A}-\vec{B}+\vec{C}$ is:

a 12 b 13
c 15 d not fixed by the magnitudes alone

Check 17

10. Two equal forces act at a point at 90° to each other. Their resultant is $20\sqrt{2}$ N. Each force is:

a 40 N b 20 N
c 80 N d 30 N

Check 18

11. Two forces of 3 N and 2 N at an angle θ have a resultant R. When the first is increased to 6 N, the resultant becomes 2R. θ is:

a 30° b 45°
c 60° d 120°

Check 19

12. The angle between $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$ is:

a 0° b 60°
c 90° d 180°

Check 20

13. A vector is $3\hat{i}+4\hat{j}+5\hat{k}$. The length of its projection on the YZ plane is:

a 5 b $\sqrt{41}$
c $\sqrt{14}$ d $\sqrt{26}$

Check 21

14. The vectors $\hat{i}+\hat{j}-\hat{k}$ and $2\hat{i}+2\hat{j}+\lambda\hat{k}$ are parallel. λ is:

The textbook gives the second vector as $2\hat{i}+2\hat{j}-\lambda\hat{k}$, which is parallel only for $\lambda=+2$, not among its options (−1, −2, −3, −4): the book's numbers don't work. With $+\lambda\hat{k}$, they do.

a −1 b −2
c −3 d −4

Check 22

15. $\vec{A}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{B}=-\hat{i}+3\hat{j}+4\hat{k}$. The projection (component) of $\vec{A}$ on $\vec{B}$ is:

a $3/\sqrt{13}$ b $3/\sqrt{26}$
c $\sqrt{3}/26$ d $\sqrt{3}/13$

The three most useful unit vectors point along the axes: $\hat{i}$ along X, $\hat{j}$ along Y and $\hat{k}$ along Z.

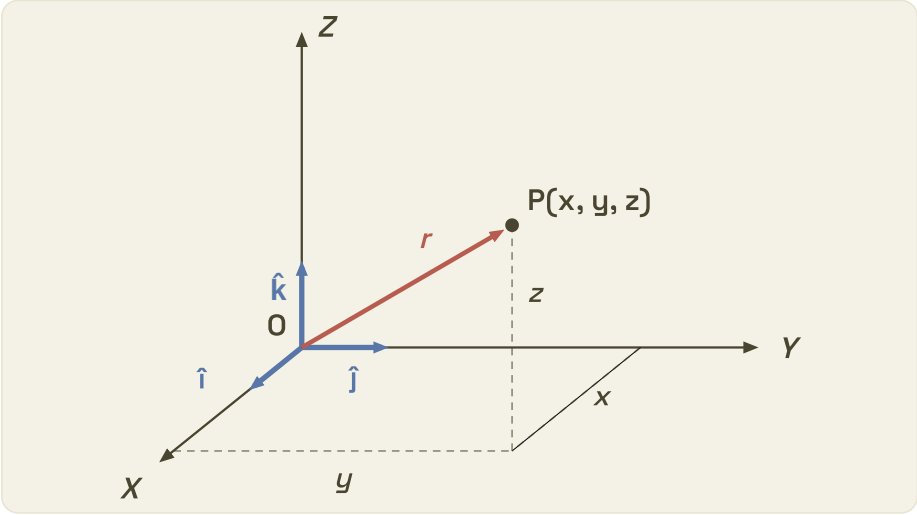


Figure 2.3 *The unit vectors î, ĵ and k̂ along the X, Y and Z axes, and the position vector r of a point P(x, y, z): x along X, then y along Y, then z up.*

The **position vector** of a point P(x, y, z) is the arrow from the origin O to P:

$$\vec{r}=\overrightarrow{OP}=x\hat{i}+y\hat{j}+z\hat{k}\equiv(x,y,z)$$

Its magnitude comes from Pythagoras, twice (across the floor, then up):

$$r=|\overrightarrow{OP}|=\sqrt{x^2+y^2+z^2}$$

so the unit vector along $\overrightarrow{OP}$ is

$$\hat{r}=\frac{\vec{r}}{r}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

WORKED EXAMPLE

A unit vector

Find the unit vector along $2\hat{i}+3\hat{j}-4\hat{k}$.

Answer

Magnitude: $\sqrt{2^2+3^2+(-4)^2}=\sqrt{4+9+16}=\sqrt{29}$.

$$\hat{n}=\frac{2\hat{i}+3\hat{j}-4\hat{k}}{\sqrt{29}}=\frac{2}{\sqrt{29}}\hat{i}+\frac{3}{\sqrt{29}}\hat{j}-\frac{4}{\sqrt{29}}\hat{k}$$

Check: $\frac{4+9+16}{29}=1$, so its length is 1.

Parallel, equal and negative vectors

- **Parallel vectors** point the same way, whatever their magnitudes (a and b in a. below). Vectors pointing exactly opposite ways are **antiparallel**.
- **Equal vectors** have the same magnitude **and** the same direction. Two cars driving north at 60 km/h on different roads have equal velocities.
- The **negative** of a vector $\vec{A}$, written $-\vec{A}$, has the same magnitude and the **opposite** direction.

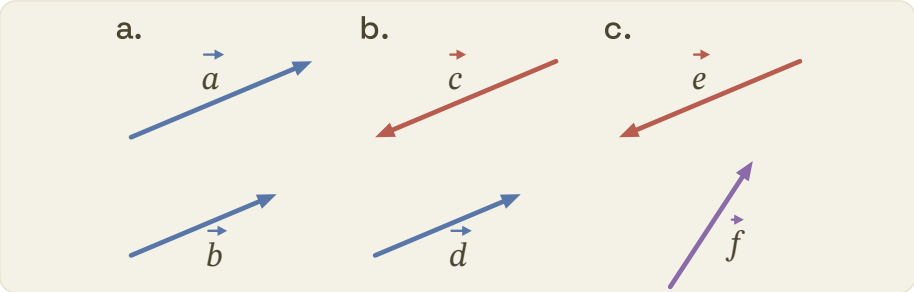


Figure 2.4 *a. Parallel vectors: the same direction. b. Antiparallel vectors: opposite directions. c. Neither.*

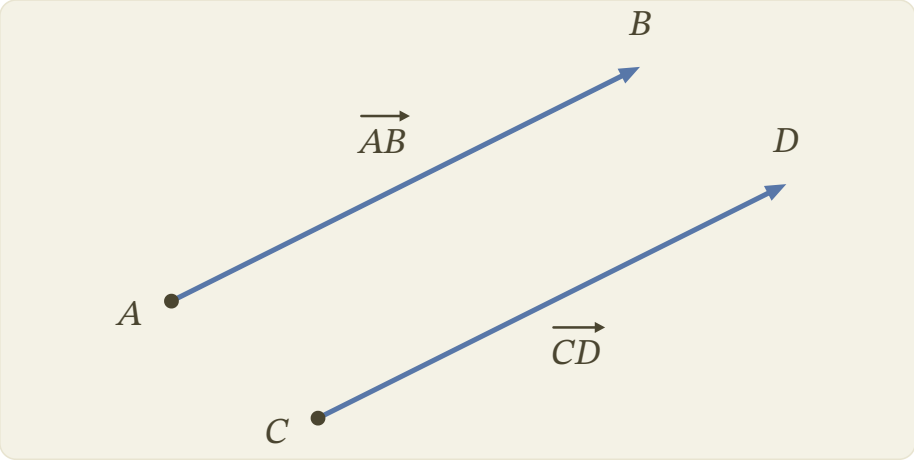


Figure 2.5 *Equal vectors: AB and CD have the same length and direction.*

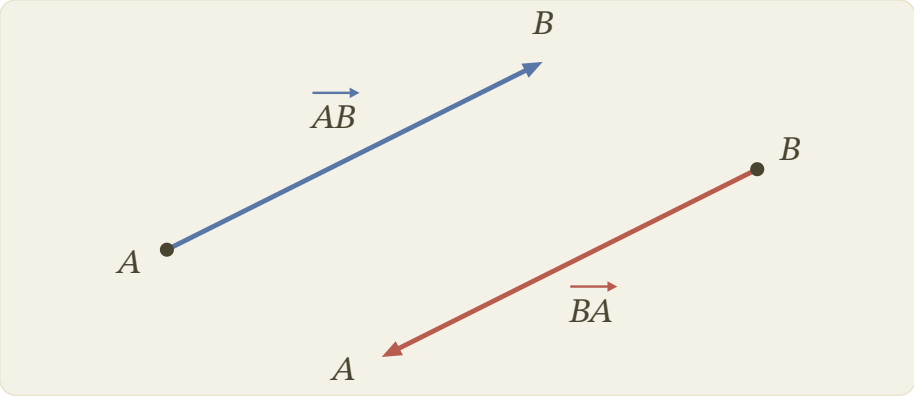


Figure 2.6 *Negative vectors: BA is the negative of AB.*

Collinear, coplanar and null vectors

- **Collinear vectors** lie along the same line, or along parallel lines, pointing the same way or opposite ways. (So parallel and antiparallel vectors are collinear.)

Exercise

Choose the correct option

Check 8

1. Two vectors of equal magnitude act at an angle θ, and α is the angle their resultant makes with one of them. Then:

a $\alpha = \theta/2$ b $\theta = \alpha/2$
c $\cos(\alpha/2) = \cos(\theta/2)$ d $\cos(\alpha/2) = ½\cos(\theta/2)$

Check 9

2. The cross product of two parallel vectors is:

a parallel to one of the vectors b perpendicular to one of the vectors
c a zero vector d a unit vector

Check 10

3. Two forces of 4 N and 3 N act on a particle at the same time. The net force on it is:

a 1 N b 7 N
c 5 N d between 1 N and 7 N, inclusive

Check 11

4. A particle's position is (3, 2, 5). Its position vector is:

a $3\hat{i} + 5\hat{j} + 2\hat{k}$ b $3\hat{i} + 2\hat{j} + 5\hat{k}$
c $5\hat{i} + 3\hat{j} + 2\hat{k}$ d $2\hat{i} + 3\hat{j} + 5\hat{k}$

Check 12

5. The vector $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ is:

The textbook prints the vector as $12\hat{i} + 12\hat{j}$ (perhaps $½\hat{i} + ½\hat{j}$). Its magnitude is then $12\sqrt{2}$ (or 0.707), so it is none of the options: the book's numbers don't work. With $1/\sqrt{2}$ for each number, it is a unit vector.

a a unit vector b a null vector
c a vector of magnitude 2 d a scalar

Check 13

6. Which of these is a vector quantity?

a electrostatic energy b electric field intensity
c electrostatic potential d electric current

Check 14

7. In a hall 10 m × 12 m × 14 m, a fly flies straight from one corner to the opposite corner. How far does it fly?

a 17 m b 21 m
c 26 m d 36 m

Answer these questions

19. a. Define the negative of a vector. State the triangle law of vector addition.

IN SHORT

Key ideas of Unit 2

- A **scalar** has only magnitude; a **vector** has magnitude and direction **and** adds by the triangle law (current, though it has a direction, is a scalar).
- **Unit vector** $\hat{n} = \vec{A}/A$; position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = \sqrt{x^2 + y^2 + z^2}$.
- **Triangle and parallelogram laws:** $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$, $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$; R lies between $|P - Q|$ and $P + Q$. **Polygon law** for many vectors; subtracting adds the negative.
- **Components:** $A_x = A\cos\theta$, $A_y = A\sin\theta$; $A = \sqrt{A_x^2 + A_y^2}$, $\tan\theta = A_y/A_x$. Add vectors by adding components.
- **Dot product** $\vec{A}\cdot\vec{B} = AB\cos\theta$, a scalar; zero when perpendicular. **Cross product** $\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$, a vector by the right-hand rule; zero when parallel; $\vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$; its magnitude is a parallelogram's area.

Project work

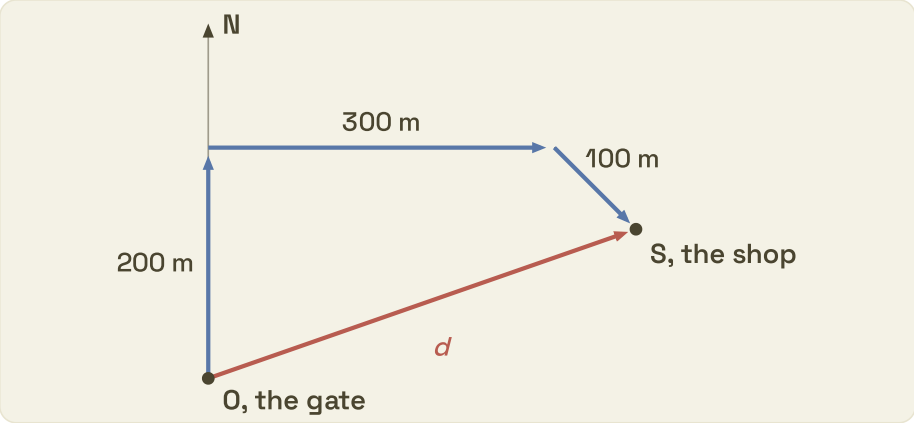


Figure 2.15 *The project's route: 200 m north, 300 m east and 100 m south-east, and the displacement d from the gate to the shop.*

- **Coplanar vectors** lie in the same plane. Any two vectors are coplanar; three need not be.
- A **null** (zero) **vector**, $\vec{0}$, has zero magnitude, and no particular direction. You get it by adding a vector to its negative: $\vec{A} + (-\vec{A}) = \vec{0}$. The displacement of a runner who finishes a lap where she started is a null vector.

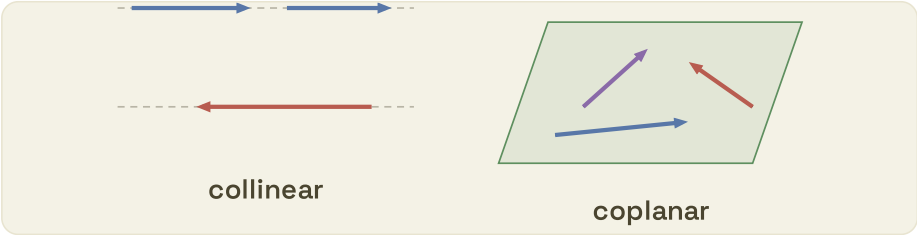


Figure 2.7 *Left: collinear vectors, on one line or on parallel lines. Right: coplanar vectors, all in one plane.*

Check 2

A runner runs once round a 400 m track and stops where she started. Her displacement is:

a a unit vector
b a null vector
c 400 m along the track
d a negative vector

2.3 Addition of vectors

Two forces on a body act together like one force, their **resultant**. Finding the single vector that has the same effect as several vectors together is called the **composition** (addition) of vectors. Since direction matters, 3 N + 4 N needn't be 7 N: it is anything from 1 N to 7 N, depending on the angle between them. Three laws, all the same idea, do the adding.

2.3.1 Triangle law of vector addition

If two vectors are represented in magnitude and direction by two sides of a triangle taken in order, their resultant is represented by the third side, taken in the opposite order.

"In order" means **head to tail**: the second vector starts where the first ends. Walk 3 m east and then 4 m north, and you end up 5 m from where you started, along the third side. That third side, from the start of the first to the end of the second, is the resultant.

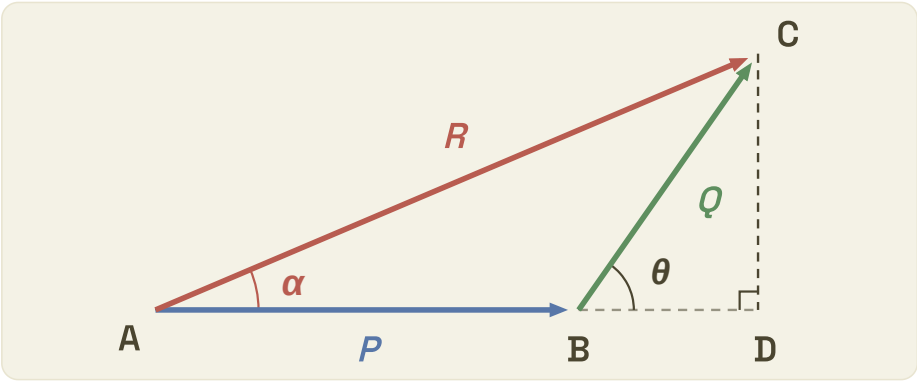


Figure 2.8 *The triangle law: AB = P, then BC = Q, so AC = R = P + Q. CD drops at a right angle onto AB produced.*

Let $\overrightarrow{AB} = \vec{P}$ and $\overrightarrow{BC} = \vec{Q}$, with θ the angle between the two vectors (between AB produced and BC, Figure 2.8). Then

$$\overrightarrow{AC} = \vec{R} = \vec{P} + \vec{Q}$$

Magnitude of R. Draw CD perpendicular to AB produced. In the right-angled triangle BDC:

$$BD = BC\cos\theta = Q\cos\theta, \qquad CD = BC\sin\theta = Q\sin\theta$$

In the right-angled triangle ADC, by Pythagoras:

$$AC^2 = AD^2 + CD^2 = (AB + BD)^2 + CD^2$$
$$R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$$
$$R^2 = P^2 + 2PQ\cos\theta + Q^2\cos^2\theta + Q^2\sin^2\theta$$

Since $\cos^2\theta + \sin^2\theta = 1$:

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

Direction of R. Let α be the angle R makes with P. In triangle ADC:

$$\tan\alpha = \frac{CD}{AD} = \frac{CD}{AB + BD} = \frac{Q\sin\theta}{P + Q\cos\theta}, \qquad \alpha = \tan^{-1}\left(\frac{Q\sin\theta}{P + Q\cos\theta}\right)$$

CAREFUL

Which θ?

θ is the angle **between the two vectors** when they start from the same point, not the angle inside the triangle at B. Inside the triangle, the angle at B is 180° − θ. Using the inside angle in $R^2 = P^2 + Q^2 + 2PQ\cos\theta$ flips the sign of the last term and gives a wrong answer.

2.3.2 Parallelogram law of vector addition

If two vectors acting at a point are represented in magnitude and direction by the two adjacent sides of a parallelogram drawn from that point, their resultant is represented by the diagonal of the parallelogram through that point.

This is the triangle law again: the side AC of the parallelogram is equal to OB (same length, same direction), so going along OA and then AC is going along P and then Q. It is the natural picture when two forces pull on one point, as two people pulling a cart with two ropes.

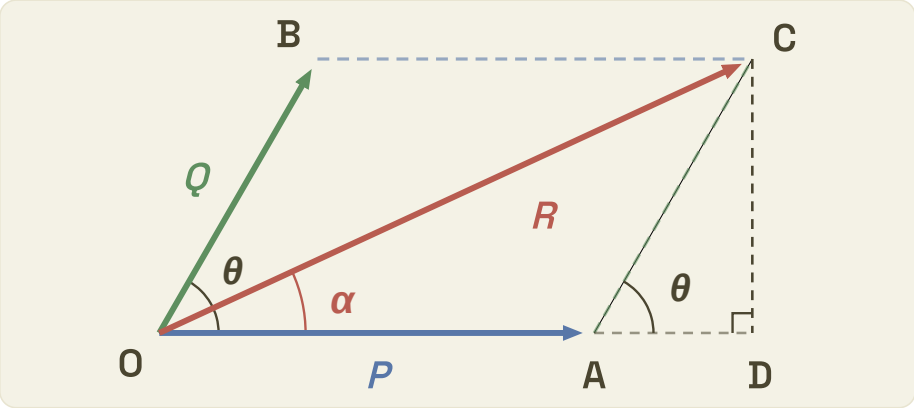


Figure 2.9 *The parallelogram law: OA = P and OB = Q from the point O; the diagonal OC is R = P + Q. AC is equal to OB, and CD drops at a right angle onto OA produced.*

CAREFUL

The two products side by side

	Scalar (dot) product	Vector (cross) product
Written	$\vec{A}\cdot\vec{B} = AB\cos\theta$	$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$
Result	a scalar	a vector, perpendicular to both
Order	$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$	$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$
Zero when (non-zero vectors)	perpendicular	parallel or antiparallel
Unit vectors	$\hat{i}\cdot\hat{i} = 1, \hat{i}\cdot\hat{j} = 0$	$\hat{i}\times\hat{i} = \vec{0}, \hat{i}\times\hat{j} = \hat{k}$
Example	work $W = \vec{F}\cdot\vec{s}$	torque $\vec{\tau} = \vec{r}\times\vec{F}$

WORKED EXAMPLE

A vector crossed with itself

Show that $\vec{A}\times\vec{A} = \vec{0}$ for $\vec{A} = 3\hat{i} + 4\hat{j} - 5\hat{k}$.

Answer

Multiply term by term, using $\hat{i}\times\hat{i} = \vec{0}$ and the cycle:

$$(3\hat{i} + 4\hat{j} - 5\hat{k}) \times (3\hat{i} + 4\hat{j} - 5\hat{k})$$
$$= 12\hat{k} + 15\hat{j} - 12\hat{k} - 20\hat{i} - 15\hat{j} + 20\hat{i} = \vec{0}$$

($3\hat{i}\times 4\hat{j} = 12\hat{k}, 3\hat{i}\times(-5\hat{k}) = 15\hat{j}, 4\hat{j}\times 3\hat{i} = -12\hat{k}, 4\hat{j}\times(-5\hat{k}) = -20\hat{i}, -5\hat{k}\times 3\hat{i} = -15\hat{j}, -5\hat{k}\times 4\hat{j} = 20\hat{i}$.) The angle between a vector and itself is 0°, and $\sin 0° = 0$.

WORKED EXAMPLE

Can both products be zero?

If $\vec{A}$ and $\vec{B}$ are non-zero vectors, can $\vec{A}\times\vec{B}$ and $\vec{A}\cdot\vec{B}$ both be zero?

Answer

No. $\vec{A}\times\vec{B} = \vec{0}$ needs $\sin\theta = 0$ ($\theta = 0°$ or 180°: parallel). $\vec{A}\cdot\vec{B} = 0$ needs $\cos\theta = 0$ ($\theta = 90°$: perpendicular). No angle has both $\sin\theta$ and $\cos\theta$ zero (since $\sin^2\theta + \cos^2\theta = 1$), so for non-zero vectors at most one of the products can vanish.

YOUR TURN

Test yourself

a. Show that the magnitude of a cross product equals the area of the parallelogram formed by the two vectors.

b. What is the angle between $(\vec{A} + \vec{B})$ and $\vec{A}\times\vec{B}$?

Answer

a. Draw the parallelogram with sides $\vec{A}$ (the base) and $\vec{B}$, at an angle θ. Its height, measured at right angles to the base, is $B\sin\theta$. Area = base × height = $A \times B\sin\theta = |\vec{A}\times\vec{B}|$.

b. **90°**. $\vec{A}\times\vec{B}$ is perpendicular to the plane of A and B, and $\vec{A} + \vec{B}$ lies in that plane. (Check: $(\vec{A} + \vec{B})\cdot(\vec{A}\times\vec{B}) = 0$, because $\vec{A}\times\vec{B}$ is perpendicular to both A and B.)

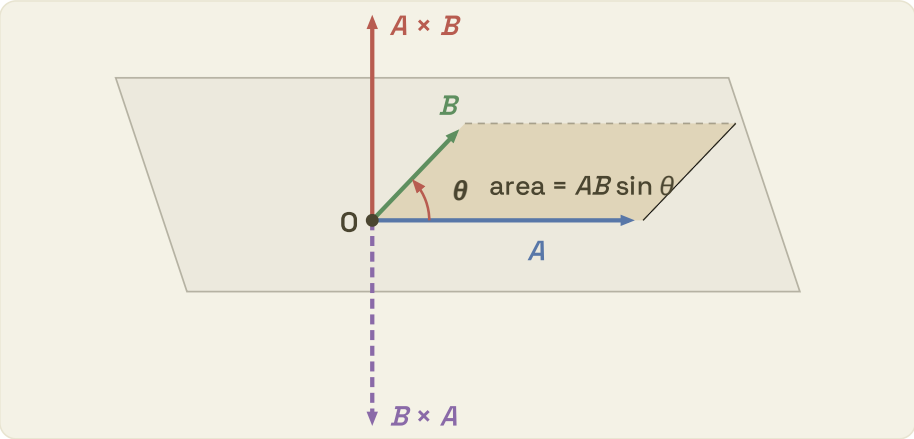


Figure 2.14 *A × B is perpendicular to the plane of A and B, the way a screw moves when turned from A towards B; B × A points the other way. Its magnitude AB sin θ is the area of the parallelogram on A and B.*

- It is **not commutative**: turning from B to A is the opposite turn, so $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$ (it is **anticommutative**).
- It is **distributive**: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$.
- Its magnitude $AB \sin\theta$ is the **area of the parallelogram** with sides $\vec{A}$ and $\vec{B}$: the base is A and the height is $B \sin\theta$.
- For the unit vectors along the axes (right-handed axes):

$$\hat{i} \times \hat{j} = \hat{k}, \qquad \hat{j} \times \hat{k} = \hat{i}, \qquad \hat{k} \times \hat{i} = \hat{j}$$
$$\hat{j} \times \hat{i} = -\hat{k}, \qquad \hat{k} \times \hat{j} = -\hat{i}, \qquad \hat{i} \times \hat{k} = -\hat{j}$$
$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

Going round the cycle i → j → k → i gives +; going backwards gives −.

In components, the cross product works out to

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\,\hat{i} + (A_z B_x - A_x B_z)\,\hat{j} + (A_x B_y - A_y B_x)\,\hat{k}$$

which is the determinant

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Check 7

$\hat{k} \times \hat{j}$ is:

a $\hat{i}$ b $-\hat{i}$

c $\vec{0}$ d 1

Proof. Let $\overrightarrow{OA} = \vec{P}$ and $\overrightarrow{OB} = \vec{Q}$ be two adjacent sides of the parallelogram OACB, with θ the angle between them. The diagonal $\overrightarrow{OC} = \vec{R}$. Draw CD perpendicular to OA produced.

Since AC is parallel and equal to OB, AC = Q and ∠CAD = θ. In triangle ACD:

$$AD = Q\cos\theta, \qquad CD = Q\sin\theta$$

In triangle OCD, by Pythagoras:

$$OC^2 = OD^2 + CD^2 = (OA + AD)^2 + CD^2$$
$$R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2 = P^2 + Q^2 + 2PQ\cos\theta$$
$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

and, if α is the angle between R and P,

$$\tan\alpha = \frac{CD}{OD} = \frac{Q\sin\theta}{P + Q\cos\theta}$$

the same results as the triangle law.

Special cases

1. **θ = 0°** (same direction): cos 0° = 1, so $R = \sqrt{P^2 + Q^2 + 2PQ} = P + Q$, and $\tan\alpha = 0$, α = 0°. The resultant is the **largest** possible, along both.
2. **θ = 90°** (at right angles): cos 90° = 0, so $R = \sqrt{P^2 + Q^2}$, and $\tan\alpha = Q/P$. (α is 45° only when P = Q.)
3. **θ = 180°** (opposite directions): cos 180° = −1, so $R = \sqrt{(P - Q)^2} = |P - Q|$, the **smallest** possible, along the larger vector.

So whatever the angle, the resultant of P and Q lies between $|P - Q|$ and $P + Q$. Two forces of 3 N and 4 N give anything from 1 N to 7 N.

Two equal vectors at 120°. With P = Q and cos 120° = −½,

$$R = \sqrt{P^2 + P^2 + 2P^2(-\tfrac{1}{2})} = \sqrt{P^2} = P$$

The resultant is as large as either vector. And the other way round: if two equal vectors have a resultant equal to each, the angle between them is 120° (next example).

WORKED EXAMPLE

Equal to either vector

Two vectors of equal magnitude have a resultant equal in magnitude to either of them. Find the angle between them.

Answer

Let P = Q = R. Then $R^2 = P^2 + Q^2 + 2PQ\cos\theta$ becomes

$$R^2 = R^2 + R^2 + 2R^2\cos\theta = 2R^2(1 + \cos\theta)$$
$$1 + \cos\theta = \tfrac{1}{2}, \qquad \cos\theta = -\tfrac{1}{2}, \qquad \theta = 120^\circ$$

WORKED EXAMPLE

The resultant of equal vectors bisects the angle

Two vectors have equal magnitudes and an angle θ between them. Show that their resultant bisects θ.

Answer

With P = Q,

$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta} = \frac{\sin\theta}{1+\cos\theta}$$

Using $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ and $1+\cos\theta = 2\cos^2\frac{\theta}{2}$:

$$\tan\alpha = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}, \qquad \alpha = \frac{\theta}{2}$$

The parallelogram on two equal sides is a rhombus, and a rhombus's diagonal bisects its angle.

YOUR TURN

Equal vectors at 120°

Two vectors of equal magnitude act at an angle of 120°. Show that their resultant equals either vector.

Answer

With P = Q and θ = 120°: $R^2 = P^2 + P^2 + 2P^2\cos 120^\circ = 2P^2 - P^2 = P^2$, so R = P = Q. (And it bisects the angle: it is at 60° to each.)

DID YOU KNOW?

The wind and the plane

A pilot flying from Kathmandu to Lukla can't simply point the plane at Lukla. The plane moves through the air, and the air itself moves over the ground: its velocity over the ground is its velocity through the air **plus** the wind's velocity, added by the parallelogram law. In a crosswind the pilot points the nose a little into the wind, so that the sum, the resultant, points at the runway. Birds do the same: a gull flying across a wind faces partly into it.

2.3.3 Polygon law of vector addition

If several vectors are represented in magnitude and direction by the sides of a polygon taken in order, their resultant is represented by the closing side of the polygon, taken in the opposite order (from the start of the first vector to the end of the last).

For four vectors drawn head to tail as $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}$ (Figure 2.10), the resultant is the closing side $\overrightarrow{AE}$: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$. It is the triangle law used again and again: add the first two, then add the third to their sum, and so on.

If the polygon closes by itself, so that the last vector ends where the first began, the resultant is the null vector: the forces balance. This is how an engineer checks that the forces on a bridge's joint balance.

- It is **commutative**: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.
- It is **distributive**: $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$.
- For the unit vectors along the axes, which are at right angles to one another:

$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1, \qquad \hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$$

So in components, the dot product is the sum of the products of matching components:

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

and the angle between two vectors is $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB}$.

WORKED EXAMPLE

Perpendicular vectors

Show that $\vec{A} = 3\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$ and $\vec{B} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ are perpendicular.

Answer

$$\vec{A}\cdot\vec{B} = (3)(3) + (4)(4) + (-5)(5) = 9 + 16 - 25 = 0$$

Both vectors are non-zero, so the dot product is zero only because cos θ = 0: they are perpendicular.

WORKED EXAMPLE

When the magnitudes subtract

Two vectors satisfy $\vec{A} - \vec{B} = \vec{C}$ and also C = A − B (for their magnitudes). Find the angle between $\vec{A}$ and $\vec{B}$.

Answer

Dot each side of $\vec{C} = \vec{A} - \vec{B}$ with itself:

$$C^2 = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B}) = A^2 + B^2 - 2AB\cos\theta$$

But $C = A - B$, so $C^2 = A^2 + B^2 - 2AB$. Comparing: $2AB\cos\theta = 2AB$, so cos θ = 1 and **θ = 0°**. Magnitudes subtract like plain numbers only when the vectors point the same way.

Check 6

A 10 N force moves a box 2 m, at 60° to the direction of motion. The work done is:

a 20 J b 10 J
c 17.3 J d 0 J

2.6.2 Vector product

The **vector product** (cross product) of $\vec{A}$ and $\vec{B}$, read "A cross B", is

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$$

where θ is the angle between them (0° to 180°) and $\hat{n}$ is the unit vector **perpendicular to the plane** of $\vec{A}$ and $\vec{B}$, in the direction given by the **right-hand rule**: curl the fingers of your right hand from $\vec{A}$ towards $\vec{B}$ (through the smaller angle), and your thumb points along $\vec{A}\times\vec{B}$. It is the way a screw moves when turned from $\vec{A}$ to $\vec{B}$.

Examples in physics: the torque of a force, $\vec{\tau} = \vec{r}\times\vec{F}$; angular momentum, $\vec{L} = \vec{r}\times m\vec{v}$; and the magnetic force on a moving charge, $\vec{F} = q\,\vec{v}\times\vec{B}$, which is why a charged particle in a magnetic field goes round in circles.

Properties:

- If θ = 0° or 180°, sin θ = 0, so $\vec{A}\times\vec{B} = \vec{0}$. For non-zero vectors, **a zero cross product means they are parallel** (or antiparallel): the property of **parallelism**. In particular $\vec{A}\times\vec{A} = \vec{0}$.
- If θ = 90°, $|\vec{A}\times\vec{B}| = AB$, the largest.

WORKED EXAMPLE

The components of a force

A force of 100 N acts at 45° to the X-axis. Find its x- and y-components.

Answer

$$F_x = F\cos\theta = 100 \times \cos 45^\circ = 100 \times \frac{1}{\sqrt{2}} = 70.7 \text{ N}$$

$$F_y = F\sin\theta = 100 \times \sin 45^\circ = 70.7 \text{ N}$$

At 45° the two components are equal. Check: $\sqrt{70.7^2 + 70.7^2} = 100$ N.

YOUR TURN

The y-component

A vector is $\vec{A} = 3\hat{i} + 4\hat{j} - 5\hat{k}$. What is the magnitude of its y-component?

Answer

The number in front of $\hat{j}$: **4**. (Its y-component is the vector $4\hat{j}$.)

Check 5

A force of 20 N acts at 30° above the horizontal. Its horizontal and vertical components are:

a 17.3 N and 10 N
b 10 N and 17.3 N
c 20 N and 0 N
d 14.1 N and 14.1 N

2.6 Scalar and vector products

Two vectors can be multiplied in two ways, and nature uses both.

The two products answer two different questions:

- The **scalar product** (dot product) asks: **how much of one vector lies along the other?** Its answer is a scalar. Work is the clearest example: only the part of a force along the motion does work.
- The **vector product** (cross product) asks: **how much of one is at right angles to the other, and which way does the pair turn?** Its answer is a vector, perpendicular to both. Torque, the turning effect of a force, is the clearest example: only the part of the force at right angles to the spanner turns the nut.

2.6.1 Scalar product

The **scalar product** (dot product) of two vectors $\vec{A}$ and $\vec{B}$, read "A dot B", is

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

where θ is the angle between them. It is a scalar. It is A times the component of B along A, or B times the component of A along B.

The work done by a force $\vec{F}$ through a displacement $\vec{s}$ is

$$W = \vec{F} \cdot \vec{s} = Fs\cos\theta$$

Push a box along the floor with a force at 30° below the horizontal: only $F\cos 30^\circ$ moves it along, and the work is $Fs\cos 30^\circ$.

Properties:

- If θ = 90°, $\vec{A} \cdot \vec{B} = AB\cos 90^\circ = 0$. For non-zero vectors, **a zero dot product means they are perpendicular** (orthogonal): the property of **orthogonality**, the test for perpendicular vectors.
- If θ = 0°, $\vec{A} \cdot \vec{B} = AB$, the largest; and $\vec{A} \cdot \vec{A} = A^2$.

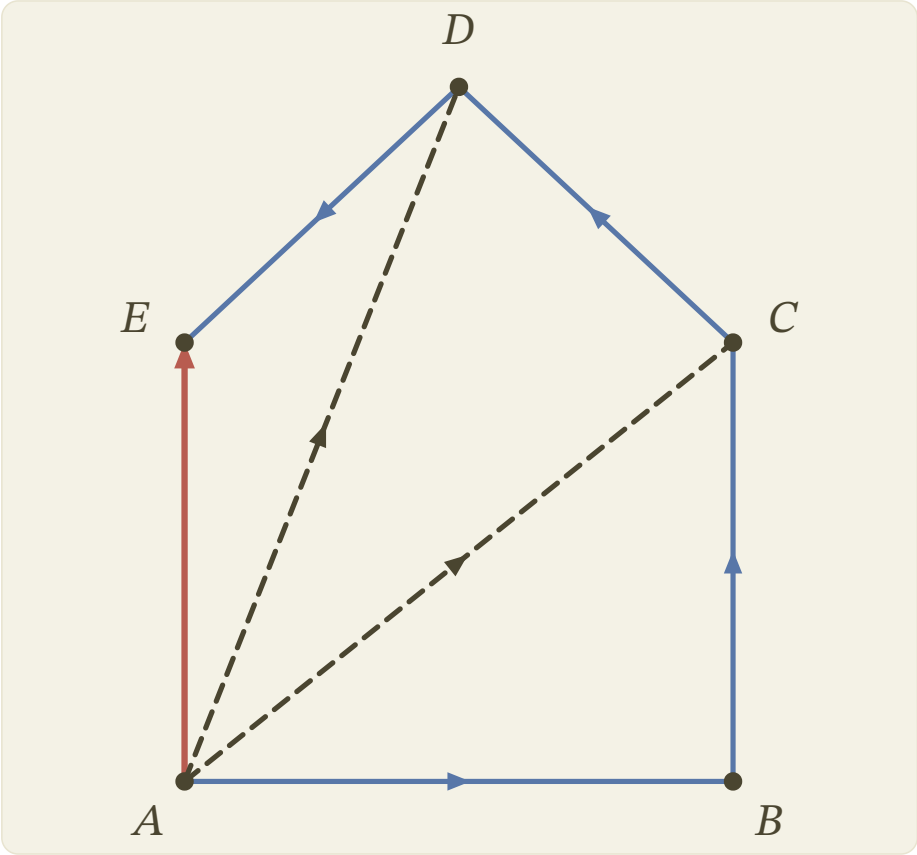


Figure 2.10 *The polygon law: the four arrows around the house shape, from A to B, B to C, C to D and D to E, add up to the closing side AE.*

Check 3

Three forces, of 3 N, 4 N and 5 N, act on a body that stays at rest. Their vectors, drawn head to tail, make:

a a closed triangle
b a straight line 12 N long
c a triangle with a 12 N closing side
d an open polygon

2.4 Subtraction of vectors

To **subtract** $\vec{b}$ from $\vec{a}$, **add the negative** of $\vec{b}$:

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

Turn $\vec{b}$ round (same length, opposite direction) to get $-\vec{b}$, then add it to $\vec{a}$ by the triangle law.

In the parallelogram on $\vec{a}$ and $\vec{b}$, the two diagonals are $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$. Subtraction also gives the **change** in a vector: if a ball's velocity changes from $\vec{u}$ to $\vec{v}$, the change is $\vec{v} - \vec{u}$, and it points along the push the ball was given.

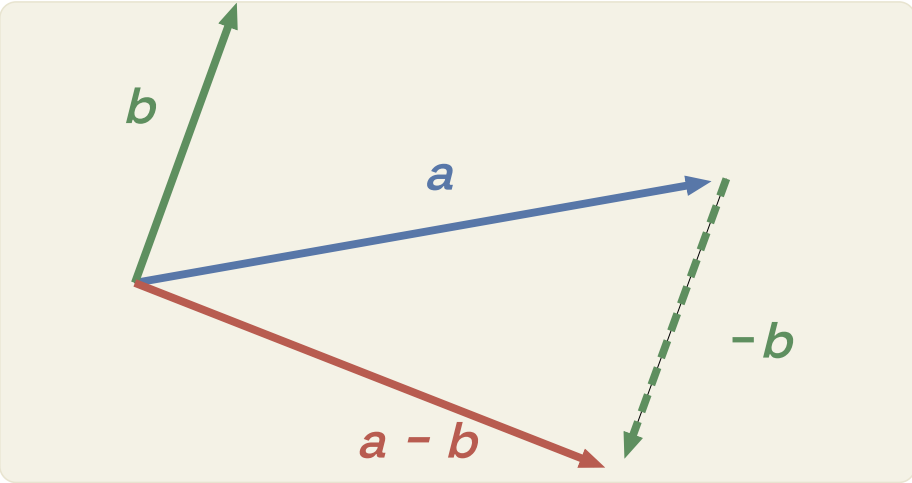


Figure 2.11 Subtraction: $-b$ drawn from the head of a; the arrow from a's tail to the head of $-b$ is $a - b$.

Check 4

A ball hits a wall at 5 m/s and bounces straight back at 5 m/s. The change in its velocity is:

a zero, since the speed is the same
b 5 m/s, away from the wall
c 10 m/s, away from the wall
d 10 m/s, towards the wall

2.5 Resolution of a vector

Resolution is the reverse of addition: a single vector is split into two (or more) vectors, its **components**, which add up to it. A component is the part of the vector that acts **along** a chosen direction, its effect in that direction, like the shadow it casts on that line.

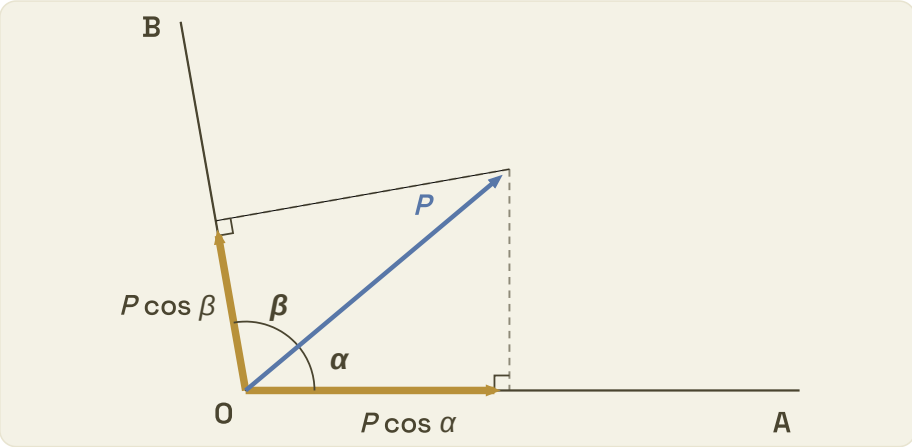


Figure 2.12 The component of P along a line is its shadow on that line: P cos α along OA, and P cos β along OB.

If a vector $\vec{P}$ makes an angle α with the direction OA and β with the direction OB, its component along OA is $P\cos\alpha$, and along OB is $P\cos\beta$. A component is the vector's magnitude times the cosine of the angle between the vector and that direction.

2.5.1 Rectangular components of a vector

The most useful directions are two at **right angles**, along the X and Y axes. Then the two components together rebuild the vector exactly.

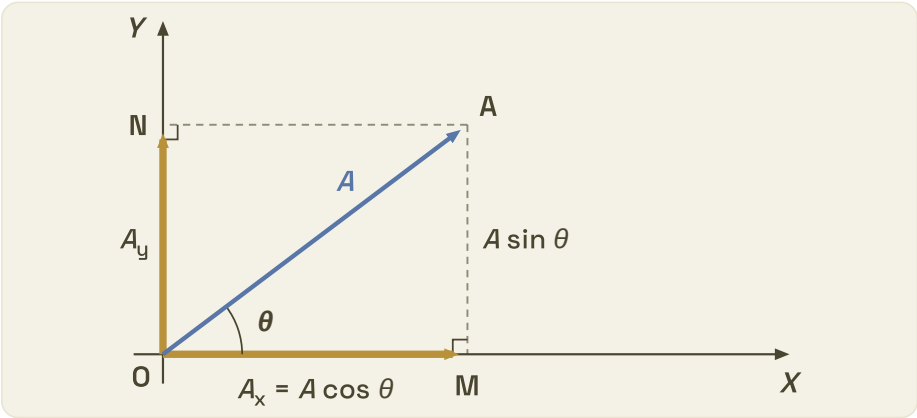


Figure 2.13 The rectangular components of A: OM = Ax along X and ON = Ay along Y.

Let $\overrightarrow{OA} = \vec{A}$ make an angle θ with the X-axis. Draw AM perpendicular to OX and AN perpendicular to OY. The **x-component** (horizontal component) is OM = A_x, and the **y-component** (vertical component) is ON = MA = A_y.

In the right-angled triangle OMA:

$$\cos\theta = \frac{OM}{OA} = \frac{A_x}{A} \quad \Rightarrow \quad A_x = A\cos\theta$$

$$\sin\theta = \frac{MA}{OA} = \frac{A_y}{A} \quad \Rightarrow \quad A_y = A\sin\theta$$

The vector is the sum of its components:

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

and the two components give back the vector's magnitude and direction:

$$A = \sqrt{A_x^2 + A_y^2}, \qquad \tan\theta = \frac{A_y}{A_x}, \qquad \theta = \tan^{-1}\frac{A_y}{A_x}$$

In space, a vector has three rectangular components, along X, Y and Z:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \equiv (A_x, A_y, A_z), \qquad A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Components make adding easy, with no triangles at all: to add vectors, **add their x-components and add their y-components** separately.