Unit 2

Answers: Vectors

Quick checks

1 c Momentum is mass × velocity, and has the velocity's direction. Speed and work have no direction; current has a direction along the wire but doesn't add by the triangle law, so it isn't a vector.

2 b She ends where she began, so her displacement has zero magnitude: a null vector. The 400 m is the distance she ran, a scalar.

3 a At rest, the resultant is zero, so the three arrows drawn head to tail must end where they began: a closed triangle (here a right-angled one, since $3^2 + 4^2 = 5^2$).

4 c Take "away from the wall" as positive: $\vec{v} - \vec{u} = (+5) - (-5) = +10$ m/s. The speed didn't change, but the velocity did, by 10 m/s, in the direction the wall pushed.

5 a $F_x = 20\cos 30^\circ = 17.3$ N and $F_y = 20\sin 30^\circ = 10$ N. Check: $\sqrt{17.3^2 + 10^2} = 20$ N. The component next to the angle takes the cosine.

6 b $W = \vec{F}\cdot\vec{s} = Fs\cos\theta = 10 \times 2 \times \cos 60^\circ = 10$ J. Only the part of the force along the motion, 5 N, does work.

7 b Going backwards round the cycle i → j → k (from k to j) gives a minus sign: $\hat{k}\times\hat{j} = -\hat{j}\times\hat{k} = -\hat{i}$. A cross product is a vector, never the number 1.

8 a For equal vectors, $\tan\alpha = \dfrac{\sin\theta}{1+\cos\theta} = \tan\dfrac{\theta}{2}$: the resultant bisects the angle.

9 c $AB\sin 0^\circ = 0$: parallel vectors have a zero cross product.

10 d The angle isn't given. The resultant can be anything from 4 − 3 = 1 N (opposite) to 4 + 3 = 7 N (together); 5 N only at right angles.

11 b The coordinates in order go with $\hat{i}, \hat{j}, \hat{k}$.

12 a Its magnitude is $\sqrt{\frac{1}{2} + \frac{1}{2}} = 1$: a unit vector, at 45° between X and Y.

13 b The electric field is the force on each coulomb of charge, so it has the force's direction. Energy and potential are scalars, and current doesn't add like a vector.

14 b The displacement is $10\hat{i} + 12\hat{j} + 14\hat{k}$ m, of magnitude $\sqrt{100 + 144 + 196} = \sqrt{440} \approx 21$ m.

15 b $\tan\theta = A_y/A_x = 1$, so θ = 45°.

16 d Without the directions, the magnitude can be anything from 12 − 3 − 4 = 5 to 3 + 4 + 12 = 19. It is 13 only if the three are mutually perpendicular: $\sqrt{9 + 16 + 144} = 13$.

17 b At 90°, $R = \sqrt{F^2 + F^2} = F\sqrt{2}$, so F = 20 N.

18 d $R^2 = 9 + 4 + 12\cos\theta$ and $4R^2 = 36 + 4 + 24\cos\theta$. So $4(13 + 12\cos\theta) = 40 + 24\cos\theta$: $52 + 48\cos\theta = 40 + 24\cos\theta$, $24\cos\theta = -12$, cos θ = –½, θ = 120°.

19 b $(\hat{i}+\hat{j})\cdot(\hat{j}+\hat{k}) = 0 + 1 + 0 = 1$, and each has magnitude $\sqrt{2}$. $\cos\theta = \dfrac{1}{\sqrt{2}\times\sqrt{2}} = \frac{1}{2}$, θ = 60°.

20 **b** In the YZ plane only the y and z parts remain, $4\hat{j} + 5\hat{k}$: $\sqrt{16 + 25} = \sqrt{41}$.

21 **b** Parallel vectors are multiples of each other. The i and j parts are doubled, so the k part must be too: $\lambda = 2 \times (-1) = -2$.

22 **b** $A\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{B} = \dfrac{-2 + 9 - 4}{\sqrt{1 + 9 + 16}} = \dfrac{3}{\sqrt{26}}$.

23 **c** $\tan\theta = 2/2 = 1$, $\theta = 45°$.

24 **b** A is perpendicular to both B and C, and the only direction perpendicular to both is that of $\vec{B} \times \vec{C}$. ($\vec{B}\cdot\vec{C}$ is a scalar, with no direction at all.)

25 **a** $|\vec{A} - \vec{B}|^2 = A^2 + B^2 - 2AB\cos\theta$, largest when $\cos\theta = -1$ ($\theta = 180°$): then it is $(A + B)^2$.

19.

a. The **negative** of a vector $\vec{A}$ is the vector $-\vec{A}$ with the same magnitude and the opposite direction. **Triangle law:** if two vectors are represented in magnitude and direction by two sides of a triangle taken in order (head to tail), their resultant is represented by the third side taken in the opposite order.

b. A **component** of a vector is the part of it acting along a given direction, its effective value in that direction: $A\cos\theta$, where θ is the angle between the vector and the direction. A vector is the sum of its components.

c. A **unit vector** has magnitude 1 and shows a direction only, $\hat{n} = \vec{A}/A$. Here $|\hat{i} + \hat{j}| = \sqrt{2}$, so

$$\hat{n} = \frac{\hat{i} + \hat{j}}{\sqrt{2}} = 0.707\,\hat{i} + 0.707\,\hat{j}$$

20.

a. If two vectors acting at a point are represented in magnitude and direction by two adjacent sides of a parallelogram drawn from that point, their resultant is represented by the diagonal through that point.

b. $AB\sin\theta = AB\cos\theta \Rightarrow \tan\theta = 1 \Rightarrow$ **$\theta = 45°$**.

c. $\tan\theta = \dfrac{3}{2} = 1.5 \Rightarrow$ **$\theta = 56.31°$**.

21.

a. **Rectangular resolution** is splitting a vector into components along mutually perpendicular directions (usually the X, Y and Z axes): $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, with $A_x = A\cos\theta$ and $A_y = A\sin\theta$ in a plane.

b. Both have magnitude 1 and lie along the Z-axis, but they point opposite ways: $\hat{i} \times \hat{j} = \hat{k}$ and $\hat{j} \times \hat{i} = -\hat{k}$. Swapping the order of a cross product reverses it.

c. $|2\hat{i} + \hat{j} - 2\hat{k}| = \sqrt{4 + 1 + 4} = 3$, so the unit vector is

$$\frac{2\hat{i} + \hat{j} - 2\hat{k}}{3} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

22. a. Yes, if $\vec{B} = -\vec{A}$: equal magnitudes and opposite directions. Then $\vec{A} + \vec{B} = \vec{0}$. (For two vectors of different magnitudes, never.)

b. Yes. The bird's velocity over the ground is the sum of its velocity through the air and the air's velocity (the wind), added by the parallelogram law. Its flapping also does this: the air pushes on each wing with a force that has an upward part (lift) and a forward part (thrust), and the bird's motion follows their resultant with gravity and drag.

c. $\sqrt{2^2 + 3^2 + 4^2} = \sqrt{4 + 9 + 16} = \sqrt{\mathbf{29}}$ **m** ≈ **5.39 m**.

23. a. No. It must also add by the triangle (or parallelogram) law. Electric current has a magnitude and a direction along the wire, but at a junction currents simply add as numbers, whatever the angle: it is a scalar.

b. If several vectors are represented in magnitude and direction by the sides of a polygon taken in order (head to tail), their resultant is represented by the closing side, from the tail of the first to the head of the last. It is the triangle law used repeatedly: add the first two, then add the third to their resultant, and so on (Figure 2.10). If the polygon closes by itself, the resultant is zero.

c. In the parallelogram with sides $\vec{A}$ and $\vec{B}$ at angle θ, take A as the base. The height is $B \sin\theta$. Area = base × height = $AB \sin\theta = |\vec{A} \times \vec{B}|$.

24. a. **In symbols:** a letter with an arrow over it, $\vec{A}$, or a bold letter **A**; its magnitude $|\vec{A}|$ or A. **In a drawing:** an arrow, whose length to scale is the magnitude and whose direction is the vector's direction, from its initial point (tail) to its terminal point (head).

b. See question 23 b.

c. With P = Q and θ = 120°, $R^2 = P^2 + P^2 + 2P^2 \cos 120^\circ = 2P^2 - P^2 = P^2$, so R = P: the resultant equals either vector.

25. a. The **scalar (dot) product**, $\vec{A} \cdot \vec{B} = AB \cos\theta$, a scalar; and the **vector (cross) product**, $\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$, a vector perpendicular to both.

b. The law, its proof and the formula $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$ are in Section 2.3.2. At θ = 180°, cos θ = −1, so $R = |P - Q|$: the resultant is the difference of the magnitudes, the smallest it can be, and points along the larger vector. (If P = Q, R = 0.)

c. A **unit vector** has magnitude 1: $\hat{n} = \vec{A}/A$. **Resolving:** with $\vec{A}$ at θ to the X-axis, drop perpendiculars from its head onto the axes; the lengths cut off are $A_x = A\cos\theta$ and $A_y = A\sin\theta$, and $\vec{A} = A_x\hat{i} + A_y\hat{j}$.

$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ because the direction comes from the right-hand rule, turning from the first vector to the second: swapping them reverses the turn, so $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$ (same magnitude, opposite direction).
Perpendicular (orthogonal): $\vec{A} \cdot \vec{B} = 0$. **Parallel:** $\vec{A} \times \vec{B} = \vec{0}$ (for non-zero vectors).

26. a. See question 19 a.

b. The derivation is in Section 2.3.1: $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$, $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$.

c. With P = A + B and Q = A − B:

$$3A^2 + B^2 = (A+B)^2 + (A-B)^2 + 2(A+B)(A-B)\cos\theta = 2A^2 + 2B^2 + 2(A^2 - B^2)\cos\theta$$
$$A^2 - B^2 = 2(A^2 - B^2)\cos\theta \quad \Rightarrow \quad \cos\theta = \tfrac{1}{2}, \qquad \theta = \mathbf{60}^\circ$$

27. $\vec{P}\cdot\vec{Q} = PQ\cos\theta = 12\cos\theta$.
a. $\cos\theta = 0$: **90°**. b. $\cos\theta = 1$: **0°**. c. $\cos\theta = -1$: **180°**.

28. a. $A = \sqrt{16+9} = \mathbf{5.00}$; $B = \sqrt{25+4} = \sqrt{29} = \mathbf{5.39}$.

b. $\vec{A} - \vec{B} = (4.00 - 5.00)\hat{i} + (3.00 + 2.00)\hat{j} = -1.00\hat{i} + 5.00\hat{j}$.

c. Magnitude $\sqrt{1+25} = \sqrt{26} = \mathbf{5.10}$. It points up and to the left (x negative, y positive): $\tan^{-1}(5/1) = 78.7^\circ$ from the −x direction, so **101.3°** from the +X-axis.

29. The reaction is 100 N at 30° to the vertical (by Newton's third law, equal and opposite to the push).
Vertical: $100\cos 30^\circ = \mathbf{86.6\ N}$. Horizontal: $100\sin 30^\circ = \mathbf{50\ N}$.
(The horizontal part is what pushes a walker forward: the ground pushes back on the foot.)

30. $5^2 = 3^2 + 4^2 + 2(3)(4)\cos\theta \Rightarrow 25 = 25 + 24\cos\theta \Rightarrow \cos\theta = 0$: A and B are perpendicular.
The triangle is right-angled, with C the hypotenuse, so $\cos\phi = \dfrac{A}{C} = \dfrac{3}{5}$, $\phi = \mathbf{53.13°}$.

31. Take east as +x and north as +y.
x: $0 - 4.75 + 0 = -4.75$ km. y: $3.25 + 0 - 1.5 = 1.75$ km.

$$d = \sqrt{4.75^2 + 1.75^2} = \sqrt{22.56 + 3.06} = \sqrt{25.63} = \mathbf{5.06\ km}$$

Direction: $\tan\phi = \dfrac{4.75}{1.75} = 2.714$, $\phi = 69.8^\circ$ from north towards west: **69.8° west of north**.

32.
$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -4 \\ 3 & -4 & 5 \end{vmatrix} = (15-16)\hat{i} - (10+12)\hat{j} + (-8-9)\hat{k} = -\hat{i} - 22\hat{j} - 17\hat{k}$$

Area $= |\vec{A}\times\vec{B}| = \sqrt{1 + 484 + 289} = \sqrt{774} = \mathbf{27.8\ square\ units}$.

33. $8^2 = 4^2 + x^2 \Rightarrow x = \sqrt{64-16} = \sqrt{48} = \mathbf{6.93\ m}$. (The displacement makes 60° with the 4 m component, since cos 60° = 4/8.)

34. Forward: $725 + 513\cos 32.4^\circ = 725 + 433.2 = 1158.2$ N. Upward: $513\sin 32.4^\circ = 274.9$ N.

$$R = \sqrt{1158.2^2 + 274.9^2} = \mathbf{1.19\times10^3\ N}$$

$$\tan\alpha = \frac{274.9}{1158.2} = 0.237, \qquad \alpha = \mathbf{13.4^\circ}$$

35. a. $\vec{A} - \vec{B} = -5.00\hat{i} + 2.00\hat{j} + 7.00\hat{k}$.

b. $A = \sqrt{4+9+16} = \sqrt{29} = \mathbf{5.39}$; $B = \sqrt{9+1+9} = \sqrt{19} = \mathbf{4.36}$.

c. $|\vec{A} - \vec{B}| = \sqrt{25+4+49} = \sqrt{78} = \mathbf{8.83}$. Yes: $\vec{B} - \vec{A} = -(\vec{A} - \vec{B})$ is the same arrow turned round, so it has the same magnitude and the opposite direction.

36. Take east as +x and north as +y. The four moves add to zero (a closed polygon), so the fourth is minus the sum of the first three.

1. 180 m west: $(-180,\ 0)$.
2. 210 m, 45° east of south: $(210\sin 45°,\ -210\cos 45°) = (148.5,\ -148.5)$.
3. 280 m, 30° east of north: $(280\sin 30°,\ 280\cos 30°) = (140.0,\ 242.5)$.

Sum: $x = -180 + 148.5 + 140.0 = 108.5$ m; $y = 0 - 148.5 + 242.5 = 94.0$ m.
The fourth displacement is $(-108.5,\ -94.0)$ m:

$$d = \sqrt{108.5^2 + 94.0^2} = \mathbf{143.6\ m}, \qquad \tan\phi = \frac{94.0}{108.5}, \quad \phi = \mathbf{40.9^\circ}$$