Unit 21 · Numerical Integration

HARDER PRACTICE QUESTIONS

Full marks 27

Write your answers on separate paper. Show your working.

1 Too big or too small, before you calculate [5]

a. Without working out any ordinates, say whether the trapezoidal rule (any n) gives too big or too small a value for each of $\int_1^3 \ln x\,dx$ and $\int_0^1 x\,e^x\,dx$. Give your reason. [2]

b. Show that the trapezoidal rule with $n = 2$, and with $n = 4$, gives the exact value of $\int_{-1}^{1} x^3\,dx$, although the graph of $y = x^3$ is not a straight line. Explain why. [2]

c. Rohit says: "For an increasing function the trapezoidal rule always gives too big a value." Use part a. to show that he is wrong. [1]

2 How many strips are enough? [5]

$\int_0^1 \dfrac{4}{1+x^2}\,dx = \pi$.

a. Find the trapezoidal values T_2 and T_4 (with 2 and 4 strips), taking the ordinates to four decimal places. [2]

b. Find the error of each, and the ratio of the two errors. Is the ratio what you expected? Why? [1]

c. Using the ratio, find the least number of strips that will make the trapezoidal error less than 0.0001. [2]

3 Working back to the ordinates [4]

Five ordinates y_0, y_1, y_2, y_3, y_4, 0.5 apart, have $y_0 = 2$ and $y_4 = 4$. The trapezoidal rule gives 6.5 for the area under them, and Simpson's rule gives $\dfrac{20}{3}$.

a. Find y_2 and $y_1 + y_3$. [3]

b. Can y_1 be found from this information? Give a reason. [1]

4 Simpson's rule from a trapezium and a rectangle [5]

On $[a, b]$, let T be the area of the single trapezium $\dfrac{b-a}{2}\left[\, f(a) + f(b) \,\right]$, and M the area of the single middle-height rectangle $(b-a)f\!\left(\dfrac{a+b}{2}\right)$ of Unit 19.

a. Show that $\dfrac{T + 2M}{3}$ is exactly Simpson's rule with $n = 2$. [2]

b. For $\int_0^2 x^3\,dx$, find T, M and the error of each. Use them to explain why Simpson's rule gives this integral exactly. [2]

c. Do the same for $\int_0^2 x^4\,dx$. Why is Simpson's rule not exact here? [1]

5 The river's flow, strip by strip [8]

A hydrologist crosses a river 12 m wide. Every 2 m from one bank she measures the depth and the water's mean speed down that line. (The numbers are made up.)

Distance from the bank (m)	0	2	4	6	8	10	12
Depth (m)	0	0.8	1.6	2.2	1.8	1.0	0
Mean speed (m/s)	0	0.4	0.9	1.2	1.0	0.6	0

a. Find the area of the cross-section by Simpson's rule. [2]

b. Estimate the flow (in m^3/s) as this area times the average of the five speeds measured in the water. [1]
c. A better way: at each line, depth × speed is the flow through a strip 1 m wide there. Work out depth × speed at each line and find the flow across the whole river by Simpson's rule. [3]
d. Which answer is bigger, and why? What single speed would make the method of b. give the answer of c.? [2]