Unit 21 · Numerical Integration

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 27

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Too big or too small, before you calculate

ANSWER HA · 5

Tests: over or under depends on which way the curve bends (f'', Unit 18), not on whether it rises; and errors of opposite sign can cancel. The tempting wrong answer is "both are increasing, so both too big".

a. $\frac{d^2}{dx^2}\ln x = -\frac{1}{x^2} < 0$: the curve bends downward, the chords lie under it, so the value is **too small**.
$\frac{d^2}{dx^2}(x\,e^x) = (x+2)e^x > 0$ on $[0, 1]$: it bends upward, so **too big**. (With $n = 2$: 1.2425 against the exact $3\ln 3 - 2 = 1.2958$, and 1.0918 against the exact 1.)

b. $n = 2$, $h = 1$, ordinates $-1, 0, 1$: $\frac{1}{2}\left[\ (-1+1) + 2(0)\ \right] = 0$. $n = 4$, $h = 0.5$, ordinates $-1, -0.125, 0, 0.125, 1$: $\frac{0.5}{2}\left[\ 0 + 2(-0.125 + 0 + 0.125)\ \right] = 0$. Exactly, $\left[\ \frac{x^4}{4}\ \right]_{-1}^{1} = 0$.

Why: on $[-1, 0]$ the curve bends downward (the trapezia are too small there) and on $[0, 1]$ it bends upward (too big). x^3 is odd, so the two halves are mirror images and the two errors are equal and opposite: they cancel.

c. $\ln x$ is increasing on $[1, 3]$, yet the trapezoidal rule gives too small a value. Rising is not what matters; bending is.

MARKING SCHEME

- ✓ $\ln x$: $f'' < 0$, bends down, too small.
- ✓ $x\,e^x$: $f'' = (x+2)e^x > 0$, bends up, too big.
- ✓ Both values worked out as 0, equal to the exact value.
- ✓ The reason: down-bending on $[-1, 0]$, up-bending on $[0, 1]$, equal and opposite errors (odd function).
- ✓ $\ln x$ rises but is underestimated, so Rohit is wrong.

2 How many strips are enough?

ANSWER A · 5

Tests: the trapezoidal error goes as h^2, so as $\frac{1}{n^2}$, and using that to plan a calculation. The tempting wrong answer in c is to think the error goes as h: "about 104 times smaller error needs about 104 times as many strips", $n \approx 416$.

a. $n = 2$: ordinates 4, 3.2, 2: $T_2 = \frac{0.5}{2}\left[\ 6 + 2(3.2)\ \right] = \mathbf{3.1000}$.

$n = 4$: ordinates 4, 3.7647, 3.2, 2.56, 2: $T_4 = \frac{0.25}{2}\left[\ 6 + 2(3.7647 + 3.2 + 2.56)\ \right] = 0.125 \times 25.0494 =$ **3.1312**.

b. Errors: $\pi - 3.1000 = 0.0416$ and $\pi - 3.1312 = 0.0104$. Ratio $\frac{0.0416}{0.0104} \approx 4$: as expected, since halving h divides the trapezoidal error by about 4 (it goes as h^2).

c. The error is about $0.0104 \times \left(\ \frac{4}{n}\ \right)^2$. For it to be less than 0.0001: $n^2 > \frac{0.0104 \times 16}{0.0001} = 1664$, so $n > 40.8$: **41 strips**. (Checked: with 40 strips the error is 0.000104, with 41 it is 0.0000991.)

MARKING SCHEME

- ✓ $T_2 = 3.1000$.
- ✓ $T_4 = 3.1312$.
- ✓ Errors 0.0416 and 0.0104, ratio about 4, because the error goes as h^2.
- ✓ Error $\approx 0.0104 \times \left(\ \frac{4}{n}\ \right)^2$ (or each doubling divides it by 4).
- ✓ $n^2 > 1664$, so 41 strips.

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3 Working back to the ordinates

ANSWER A · 4

Tests: what each rule takes from the data, written as equations and solved backwards. The tempting mistake is to treat y_1 and y_3 as two unknowns and look for a third equation that doesn't exist.

a. Let $u = y_1 + y_3$ and $v = y_2$.

Trapezoidal: $\frac{0.5}{2}\left[\ 6 + 2(u+v)\ \right] = 6.5$, so $u + v = 10$.

Simpson's: $\frac{0.5}{3}\left[\ 6 + 4u + 2v\ \right] = \frac{20}{3}$, so $4u + 2v = 34$, $2u + v = 17$.

Subtracting: $u = 7$, $v = 3$. So $\mathbf{y_2 = 3}$ and $\mathbf{y_1 + y_3 = 7}$.

b. **No.** Both rules multiply y_1 and y_3 by the same weight (2 in the trapezoidal rule, 4 in Simpson's), so they use only their sum: $y_1 = 3, y_3 = 4$ and $y_1 = 1, y_3 = 6$ give the same two answers.

MARKING SCHEME

- ✓ $u + v = 10$ from the trapezoidal rule.
- ✓ $2u + v = 17$ from Simpson's rule.
- ✓ $y_2 = 3$ and $y_1 + y_3 = 7$.
- ✓ No: both rules weight y_1 and y_3 equally, so only their sum is fixed.

4 Simpson's rule from a trapezium and a rectangle

ANSWER HA · 5

Tests: where Simpson's weights come from, seen as a weighted average of two simple rules whose errors have opposite signs. The tempting wrong idea is that Simpson's rule is "a better trapezoidal rule"; in fact it leans on the middle rectangle twice as much as on the trapezium.

a. With $h = \frac{b-a}{2}$ and $y_0 = f(a), y_1 = f\left(\ \frac{a+b}{2}\ \right), y_2 = f(b)$: $T = h(y_0 + y_2)$ and $M = 2hy_1$. So $\frac{T + 2M}{3} = \frac{h}{3}(y_0 + y_2 + 4y_1)$, Simpson's rule with $n = 2$.

b. $T = \frac{2}{2}(0 + 8) = 8$, $M = 2 \times 1 = 2$; exact $\left[\ \frac{x^4}{4}\ \right]_0^2 = 4$. Errors $4 - 8 = -4$ and $4 - 2 = +2$. The trapezium's error is exactly -2 times the rectangle's, so in $T + 2M$ they cancel: $\frac{8+4}{3} = 4$, exact.

c. $T = \frac{2}{2}(0 + 16) = 16$, $M = 2 \times 1 = 2$; exact $\frac{32}{5} = 6.4$. Errors -9.6 and $+4.4$: the ratio is not -2, so they don't cancel: $\frac{16+4}{3} = \frac{20}{3} = 6.67$, not 6.4.

MARKING SCHEME

- ✓ $T = h(y_0 + y_2)$ and $M = 2hy_1$ with $h = \frac{b-a}{2}$.
- ✓ $\frac{T + 2M}{3} = \frac{h}{3}(y_0 + 4y_1 + y_2)$.
- ✓ x^3: $T = 8$, $M = 2$, errors -4 and $+2$.
- ✓ The errors are in the ratio $-2 : 1$, so $T + 2M$ cancels them: exact 4.
- ✓ x^4: errors -9.6 and $+4.4$, not in the ratio $-2 : 1$, so $\frac{20}{3} \neq 6.4$.

5 The river's flow, strip by strip

ANSWER

Tests: integrating a product, not multiplying two averages; seeing from the data why the two differ. The tempting wrong answer is that b. and c. must agree because they use the same numbers.

a. $h = 2$: $\frac{2}{3}\left[\ (0 + 0) + 4(0.8 + 2.2 + 1.0) + 2(1.6 + 1.8)\ \right] = \frac{2}{3}(16 + 6.8) = \mathbf{15.2\ m^2}$.

b. Average speed $\frac{0.4 + 0.9 + 1.2 + 1.0 + 0.6}{5} = 0.82$ m/s. Flow $\approx 15.2 \times 0.82 = \mathbf{12.5\ m^3/s}$ (12.46).

c. Depth × speed: 0, 0.32, 1.44, 2.64, 1.80, 0.60, 0 (m^2/s). Simpson's: $\frac{2}{3}\left[\ 0 + 4(0.32 + 2.64 + 0.60) + 2(1.44 + 1.80)\ \right] = \frac{2}{3}(14.24 + 6.48) = \mathbf{13.8\ m^3/s}$ (13.81).

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d. The strip-by-strip flow is about 11% bigger. The water is fastest where the river is deepest (2.2 m at 1.2 m/s), so the deep middle carries much more than its share; a plain average of the speeds gives the slow shallow edges the same say as the fast deep middle. The speed that makes area × speed right is $\frac{13.81}{15.2} \approx$ **0.91 m/s**, the average weighted by depth, not 0.82 m/s. This is why the US Geological Survey adds the flow strip by strip.

MARKING SCHEME

- ✓ a. Simpson's rule set up with $h = 2$: 4×4.0 and 2×3.4.
- ✓ Area 15.2 m^2.
- ✓ b. $15.2 \times 0.82 \approx 12.5$ m^3/s.
- ✓ c. The products 0.32, 1.44, 2.64, 1.80, 0.60.
- ✓ Simpson's rule on them set up (4×3.56, 2×3.24).
- ✓ Flow ≈ 13.8 m^3/s.
- ✓ d. c. is bigger, because the deepest water is also the fastest, which the plain average of speeds ignores.
- ✓ The speed $13.8 \div 15.2 \approx 0.91$ m/s.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	17	10	**27**

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