Unit 21 · Numerical Integration

ANSWER KEY · FOR TEACHERS

Full marks 35

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 3

1. numerical integration
2. ordinates
3. chord
4. parabola
5. even
6. error

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: Only the odd ordinates $y_1, y_3, \ldots$ are multiplied by 4; the even ones inside, $y_2, y_4, \ldots$, by 2.

iii. ✗ **False**: It is too big where the curve bends upward (the chords lie above it) and too small where it bends downward.

iv. ✓ **True**

v. ✗ **False**: It divides the error by about 4: the error goes as h^2.

vi. ✓ **True**

Each carries 0.5 mark.

3 Multiple choice

i. **c.** $\frac{h}{3}$ — K · 1

$\int_a^b y\,dx \approx \frac{h}{3}\left[\,(y_0 + y_n) + 4(\text{odd}) + 2(\text{even})\,\right]$, hence "one-third rule". $\frac{h}{2}$ belongs to the trapezoidal rule; $\frac{h}{6}$ is Kepler's barrel rule written with the whole width $2h$ of a pair.

ii. **b.** $\frac{1}{2}$ — U · 1

7 ordinates are 6 strips, so $h = \frac{5-2}{6} = \frac{1}{2}$. $\frac{3}{7}$ divides by the number of ordinates instead of strips; 3 is a single strip.

iii. **a.** 1, 4, 2, 4, 2, 4, 1 — U · 1

The odd ordinates are the middles of the three pairs (×4); y_2 and y_4 are each the end of two pairs (×2). The last option is the trapezoidal rule's weights.

iv. **b.** too small, because sin x bends downward there and every chord lies under the curve — U · 1

$\frac{d^2}{dx^2}\sin x = -\sin x < 0$ on $(0, \frac{\pi}{2})$: the curve is concave downward, so each trapezium misses a sliver of area. (With $n = 2$ the rule gives 0.9481, with $n = 4$ 0.9871; the exact value is 1.) Increasing is not the point; bending is.

v. **c.** 13 — A · 1

$h = 1$: $\frac{1}{3}\left[\,(1+2) + 4(3+4) + 2(4)\,\right] = \frac{39}{3} = 13$. 12.5 is the trapezoidal rule; 11 swaps the weights 4 and 2; 39 forgets the $\frac{h}{3}$.

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vi. **b.** Simpson's rule with 4 strips HA · 1

With h the width of 4 strips: $T_4 = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$ and $T_2 = h[y_0 + 2y_2 + y_4]$. So $4T_4 - T_2 = h[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$, and dividing by 3 gives Simpson's weights. For $\sin x$ on $[0, \pi]$: $\frac{4(1.896119) - 1.570796}{3} = 2.004560$, the table's Simpson value.

4 Spot the error

MARKING SCHEME U · 2

- ✓ The mistake: h is the interval divided by the number of strips, 4, not of ordinates: $h = 0.5$.
- ✓ Area $= 0.25 \times 19 = 4.75$.

5 Short questions

a. **ANSWER** A · 5

Seven offsets: $n = 6$, $h = 10$ m. $y_0 + y_6 = 26$; odd $15 + 17 + 16 = 48$; even $18 + 20 = 38$.

Area $\approx \frac{10}{3}\left[\ 26 + 4(48) + 2(38)\ \right] = \frac{10}{3} \times 294 =$ **980 m²**.

$\frac{980}{508.74} = 1.926$ ropani, and $0.926 \times 16 = 14.8$ aana: about **1 ropani 15 aana**. (The trapezoidal rule gives 990 m².)

MARKING SCHEME

- ✓ Seven offsets: $n = 6$, $h = 10$ m.
- ✓ $y_0 + y_6 = 26$, odd 48, even 38.
- ✓ $\frac{10}{3}\left[\ 26 + 4(48) + 2(38)\ \right] = \frac{10}{3} \times 294$.
- ✓ Area = 980 m².
- ✓ $980 \div 508.74 = 1.93$ ropani, about 1 ropani 15 aana.

b. **ANSWER** A · 5

a. $h = 3$: $\frac{3}{2}\left[\ (0 + 0) + 2(5.2 + 7.8 + 8.4 + 6.6)\ \right] = 1.5 \times 56 =$ **84 m²**.

b. Six widths are 5 strips, an odd number, and Simpson's rule takes the strips in pairs.

c. First four strips: $\frac{3}{3}\left[\ (0 + 6.6) + 4(5.2 + 8.4) + 2(7.8)\ \right] = 6.6 + 54.4 + 15.6 = 76.6$ m². The last strip: $\frac{3}{2}(6.6 + 0) = 9.9$ m². Area $\approx 76.6 + 9.9 =$ **86.5 m²**.

MARKING SCHEME

- ✓ a. $h = 3$ and the rule set up with the four inside widths doubled.
- ✓ 84 m².
- ✓ b. 5 strips is odd; Simpson's rule needs pairs of strips.
- ✓ c. Simpson's rule on the first four strips: 76.6 m².
- ✓ Adding the last trapezium, 9.9 m²: 86.5 m².

6 Long question

ANSWER

a. Putting in the points: $y_0 = Ah^2 - Bh + C$, $y_1 = C$, $y_2 = Ah^2 + Bh + C$.

Area $= \int_{-h}^{h} (Ax^2 + Bx + C)\,dx = \left[\ \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\ \right]_{-h}^{h} = \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C)$.

Since $y_0 + y_2 = 2Ah^2 + 2C$, $2Ah^2 + 6C = (y_0 + y_2) + 4C = y_0 + 4y_1 + y_2$. So the area is $\frac{h}{3}(y_0 + 4y_1 + y_2)$.

b. $h = \frac{1.5}{6} = 0.25$. At $x = 0, 0.25, \ldots, 1.5$: $y = 1, 0.7788, 0.6065, 0.4724, 0.3679, 0.2865, 0.2231$.

$y_0 + y_6 = 1.2231$; odd $0.7788 + 0.4724 + 0.2865 = 1.5377$; even $0.6065 + 0.3679 = 0.9744$.

Simpson's: $\frac{0.25}{3}\left[\ 1.2231 + 4(1.5377) + 2(0.9744)\ \right] = \frac{9.3227}{12} =$ **0.7769**.

Trapezoidal: $\frac{0.25}{2}\left[\ 1.2231 + 2(1.5377 + 0.9744)\ \right] = 0.125 \times 6.2473 =$ **0.7809**.

c. Exactly, $\int_0^{1.5} e^{-x}\,dx = \left[\ -e^{-x}\ \right]_0^{1.5} = 1 - e^{-1.5} = 0.776870$.

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Trapezoidal error $= 0.776870 - 0.7809 = -0.0040$; Simpson's error is about -0.00002, some 200 times smaller. **A · 8**

$\frac{d^2}{dx^2}e^{-x} = e^{-x} > 0$: the curve bends upward, so every chord lies above it and the trapezia hold a little extra area.

MARKING SCHEME

- ✓ a. The three equations, with $y_1 = C$.
- ✓ The area $\frac{2Ah^3}{3} + 2Ch$ (the Bx part gives 0).
- ✓ $y_0 + y_2 = 2Ah^2 + 2C$ used to reach $\frac{h}{3}(y_0 + 4y_1 + y_2)$.
- ✓ b. $h = 0.25$ and the seven ordinates.
- ✓ Simpson's rule: 0.7769.
- ✓ Trapezoidal rule: 0.7809.
- ✓ c. $1 - e^{-1.5} = 0.7769$ (0.776870); errors -0.0040 and about -0.00002.
- ✓ Too big because e^{-x} bends upward ($f'' > 0$): the chords lie above the curve.

7 Think

ANSWER **HA · 3**

The exact area is two triangles, $2 \times \frac{1}{2} \times 2 \times 2 = 4$. With $h = 2$, the ordinates are 2, 0, 2.

Trapezoidal: $\frac{2}{2}\left[\,(2+2) + 2(0)\,\right] = 4$, exact. Simpson's: $\frac{2}{3}\left[\,(2+2) + 4(0)\,\right] = \frac{8}{3} \approx 2.67$, far off.

The graph is two straight pieces, so the chords lie exactly on it; Simpson's rule puts one smooth parabola through (0, 2), (2, 0), (4, 2), which sags below the V. So Bishal is wrong: Simpson's rule is better for smooth curves, but not where the graph (or data) has a corner inside a pair of strips.

MARKING SCHEME

- ✓ Exact value 4, and the trapezoidal rule gives 4.
- ✓ Simpson's rule gives $\frac{8}{3}$.
- ✓ The reason: the chords follow the straight pieces exactly, while one parabola can't follow a corner; Simpson's is better only for smooth curves.

8 Activity: The area of a plate

(not marked)

WHAT STUDENTS SHOULD FIND

Both rules come out a little under πr^2, Simpson's usually closer, within a few percent. The shortfall is largest at the two ends, where the circle turns steeply and a chord or parabola can't follow it; the rest of the difference is measuring error and a plate that is not a perfect circle.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	4	8	19	4	**35**

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