Unit 20 · Numerical Computation

HARDER PRACTICE QUESTIONS

Full marks 29

Write your answers on separate paper. Show your working. Angles are in radians.

1 A root with no sign change [6]

$f(x) = x^3 - 3x + 2$. Yangchen finds $f(0) = 2$ and $f(3) = 20$, and says: "Both are positive, so $f(x) = 0$ has no root between 0 and 3."

a. Factorize $f(x)$, and show that Yangchen is wrong. Could bisection ever close in on the root you found, from any interval? [2]

b. Do three steps of Newton–Raphson from $x_0 = 2$, and write the error $x_n - 1$ of each value. Do the correct decimal places double, as they did for $x = \cos x$? [2]

c. Show that for this f, $x_{n+1} - 1 = (x_n - 1)\dfrac{2x_n + 1}{3(x_n + 1)}$, and use it to explain what you saw in b. [2]

2 A cycle you can predict [6]

Sujan uses Newton–Raphson on $f(x) = x^3 - 5x$, whose roots are 0 and $\pm\sqrt{5}$.

a. Start from $x_0 = 1$. What happens? [1]

b. Find every x_0 for which $x_1 = -x_0$. Then explain why, for this f, any such start goes back and forth for ever. [3]

c. Choose a start that leads to the root $\sqrt{5}$, and find that root correct to four decimal places. [2]

3 Why the digits double [5]

For $\sqrt{N}$, Newton–Raphson on $f(x) = x^2 - N$ gives $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{N}{x_n} \right)$.

a. Show that $x_{n+1} - \sqrt{N} = \dfrac{(x_n - \sqrt{N})^2}{2x_n}$. [2]

b. Hence show that, from any start $x_0 > 0$, every later value is at least $\sqrt{N}$. [1]

c. For $\sqrt{10}$ from $x_0 = 3$, $x_1 = 3.166667$, whose error is 0.004389. Without working out x_2, say about how large its error will be, and so how many decimal places of x_2 you can trust. [2]

4 Is it correct to one decimal place? [5]

Saugat wants $\sqrt{2.1}$, the root of $x^2 - 2.1 = 0$, correct to one decimal place, by bisection on [1, 2]. He reasons: "One decimal place means an error under 0.05. After 5 steps the error is at most $\dfrac{1}{32} < 0.05$, and the fifth midpoint is 1.46875. So $\sqrt{2.1} = 1.5$."

a. Show, with one multiplication, that his answer is wrong to one decimal place. [1]

b. Where does his reasoning go wrong? [2]

c. Using the rule that both ends of the interval must round to the same value, find how many steps are really needed, and the answer. [2]

5 How deep is the water? [7]

A spherical water tank has inner radius 1 m. When the water is h meters deep, it holds $V = \dfrac{1}{3}\pi h^2(3 - h)$ cubic meters (the formula for a piece of a sphere). The tank holds 2 m^3 of water. (The tank is made up for the problem.)

a. Show that h is a root of $f(h) = h^3 - 3h^2 + \dfrac{6}{\pi} = 0$. [1]

b. Show that this equation has three real roots, find an interval of width 1 for each, and say which one is the depth. [2]

c. Karma wants to start Newton–Raphson from $h_0 = 0$ (an empty tank) or $h_0 = 2$ (a full one). Why will neither work? [1]

d. Find the depth correct to four decimal places, starting from $h_0 = 1$. [2]

e. Without the equation, explain why the depth must be a little less than 1 m. [1]