Unit 20 · Numerical Computation

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 29

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 A root with no sign change

HA · 6

ANSWER

Tests: a sign change is what bisection needs, not what a root is; and why Newton–Raphson slows down at a root where the curve only touches the axis. The tempting answers are Yangchen's, and "Newton–Raphson always doubles the digits".

a. $f(1) = 0$, so $x - 1$ is a factor: $f(x) = (x-1)^2(x+2)$. So $x = 1$ is a root in $(0, 3)$: Yangchen is wrong. At $x = 1$ the curve touches the X-axis and turns back up ($(x-1)^2 \geq 0$, and $x + 2 > 0$ near 1), so f never changes sign there. Bisection can't close in on it from any interval: it needs ends of opposite signs, and the only sign change of f is at $x = -2$, so every interval it keeps holds -2 and it closes in on -2. (Only by luck could a midpoint land exactly on 1, as from $[-3, 5]$.)

b. $f'(x) = 3x^2 - 3$. $x_1 = 2 - \frac{4}{9} = 1.555556$, $x_2 = 1.297907$, $x_3 = 1.155390$. The errors are 1, 0.556, 0.298, 0.155: each about **half** the one before. The places don't double; the error only halves, as slowly as bisection.

c. $f'(x) = 3(x-1)(x+1)$, so $\frac{f(x)}{f'(x)} = \frac{(x-1)(x+2)}{3(x+1)}$, and

$x_{n+1} - 1 = (x_n - 1) - \frac{(x_n-1)(x_n+2)}{3(x_n+1)} = (x_n - 1)\frac{3x_n + 3 - x_n - 2}{3(x_n+1)} = (x_n - 1)\frac{2x_n+1}{3(x_n+1)}$. ✓

Near $x = 1$ the fraction is close to $\frac{3}{6} = \frac{1}{2}$: each error is about half the last, not its square. The cause is that $f'(1) = 0$ as well as $f(1) = 0$: near the root the tangent is almost flat, so each tangent lands only half-way.

MARKING SCHEME

- ✓ $f(x) = (x-1)^2(x+2)$, so $x = 1$ is a root in $(0, 3)$.
- ✓ No sign change at $x = 1$: bisection can't close in on it (it goes to -2).
- ✓ $x_1 = 1.555556$, $x_2 = 1.297907$, $x_3 = 1.155390$.
- ✓ Errors 0.556, 0.298, 0.155: halving, not doubling places.
- ✓ The algebra to $(x_n - 1)\frac{2x_n+1}{3(x_n+1)}$.
- ✓ The fraction tends to $\frac{1}{2}$ because $f'(1) = 0$.

2 A cycle you can predict

ANSWER

Tests: a Newton–Raphson failure worked out from the formula and the function's symmetry, rather than recalled. The tempting conclusion in a. is an arithmetic slip, or that the method has found two roots, 1 and -1.

a. $f'(x) = 3x^2 - 5$. $x_1 = 1 - \frac{-4}{-2} = -1$, $x_2 = -1 - \frac{4}{-2} = 1$: the values go 1, -1, 1, -1, … for ever, and neither is a root ($f(1) = -4$, $f(-1) = 4$).

b. $x_1 = x_0 - \frac{{x_0}^3 - 5x_0}{3{x_0}^2 - 5} = \frac{2{x_0}^3}{3{x_0}^2 - 5}$. Setting it equal to $-x_0$: $2{x_0}^3 = -3{x_0}^3 + 5x_0$, $5{x_0}^3 = 5x_0$, so $x_0 = 0$ or ± 1. ($x_0 = 0$ is already a root and stays there.)

f is odd, so the Newton step $g(x) = \frac{2x^3}{3x^2 - 5}$ is odd too: $g(-x) = -g(x)$. If $x_1 = g(x_0) = -x_0$, then $x_2 = g(-x_0) = -g(x_0) = x_0$: back to the start, so the pair repeats for ever. Starting at ± 1 cycles.

c. Start past the turning point $x = \sqrt{5/3} \approx 1.29$ of f, on the side of $\sqrt{5}$, say $x_0 = 2$ ($f(2) = -2$, $f(3) = 12$: the root is in $(2, 3)$). $x_1 = 2 - \frac{-2}{7} = 2.285714$, $x_2 = 2.237640$, $x_3 = 2.236070$, $x_4 = 2.236068$: **2.2361**.

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MARKING SCHEME HA · 6

✓ a. 1, −1, 1, … for ever, no root reached.

✓ b. $x_1 = \dfrac{2x_0^{\,3}}{3x_0^{\,2} - 5}$.

✓ $x_1 = -x_0$ gives $x_0 = 0, \pm 1$.

✓ f odd, so g odd: $x_2 = g(-x_0) = -g(x_0) = x_0$, a cycle.

✓ c. A start beyond the turning point (e.g. 2), with a reason.

✓ 2.2361.

3 Why the digits double

ANSWER A · 5

Tests: why the correct places double, from the formula; using the error formula to predict instead of iterate. The tempting reading is that the error falls by a fixed factor each step, as in bisection.

a. $x_{n+1} - \sqrt{N} = \dfrac{x_n^{\,2} + N}{2x_n} - \sqrt{N} = \dfrac{x_n^{\,2} - 2\sqrt{N}x_n + N}{2x_n} = \dfrac{(x_n - \sqrt{N})^2}{2x_n}$. ✓

b. With $x_n > 0$, the right side is a square over a positive number, so it is ≥ 0: $x_{n+1} \geq \sqrt{N}$ for every $n \geq 0$. (So $x_1, x_2, \ldots$ are all at least $\sqrt{N}$, and all positive, whatever $x_0 > 0$ was.)

c. $\dfrac{0.004389^2}{2 \times 3.166667} \approx 0.000003$ (3.0×10^{-6}). So x_2 is within about 0.000003 of $\sqrt{10}$, and too big (b.): about **five decimal places** can be trusted. (Indeed $x_2 = 3.162281$ and $\sqrt{10} = 3.162278$.) The error went from about 10^{-3} to 10^{-6}: squared, so the places doubled.

MARKING SCHEME

✓ a. Over a common denominator, $\dfrac{x_n^{\,2} - 2\sqrt{N}x_n + N}{2x_n}$.

✓ It is $\dfrac{(x_n - \sqrt{N})^2}{2x_n}$.

✓ b. A square over a positive number is ≥ 0, so $x_{n+1} \geq \sqrt{N}$.

✓ c. Error $\approx 0.004389^2 / 6.33 \approx 3 \times 10^{-6}$.

✓ About five decimal places trusted.

4 Is it correct to one decimal place?

ANSWER HA · 5

Tests: an error bound is not the same as correct rounding; a root close to a rounding boundary needs more steps. The tempting answer is Saugat's 1.5 after 5 steps.

a. $1.45^2 = 2.1025 > 2.1$, so $\sqrt{2.1} < 1.45$, which rounds to **1.4**, not 1.5.

b. His bound is right: 1.46875 is within $\dfrac{1}{32}$ of the root. But a number within 0.05 of the root need not round the same way as the root. The root, 1.449…, is just below 1.45, where rounding to one place changes from 1.4 to 1.5, so a midpoint a little above it, 1.46875, rounds the other way.

c. The intervals: [1.4375, 1.5] after step 4, [1.4375, 1.46875] after 5, [1.4375, 1.453125] after 6, [1.4453125, 1.453125] after 7 (still across 1.45), and after step 8 ($f(1.44921875) = 0.00023 > 0$) [1.4453125, 1.44921875], whose ends both round to 1.4. So **8 steps**, and $\sqrt{2.1} = 1.4$ to one decimal place.

MARKING SCHEME

✓ a. $1.45^2 > 2.1$, so the root is below 1.45: 1.4.

✓ b. The error bound holds, but being within 0.05 doesn't fix the rounding.

✓ The root is close to 1.45, the rounding boundary.

✓ c. The intervals to step 8, [1.4453125, 1.44921875].

✓ 8 steps, 1.4.

5 How deep is the water?

ANSWER

Tests: setting up an equation from a setting, choosing the root that makes sense, and a start that avoids the turning points. The tempting starts are the natural ends, 0 and 2, where the tangent is flat.

a. $\dfrac{1}{3}\pi h^2(3 - h) = 2$ gives $\pi(3h^2 - h^3) = 6$, so $h^3 - 3h^2 + \dfrac{6}{\pi} = 0$.

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b. $\frac{6}{\pi} \approx 1.9099$. $f(-1) = -2.0901$, $f(0) = 1.9099$, $f(1) = -0.0901$, $f(2) = -2.0901$, $f(3) = 1.9099$. Sign changes in $(-1, 0)$, $(0, 1)$ and $(2, 3)$: three real roots (a cubic has at most three). The depth must be between 0 and 2, the tank's height, so it is the root in **(0, 1)**.

c. $f'(h) = 3h^2 - 6h = 3h(h - 2)$ is 0 at both $h = 0$ and $h = 2$: the tangent there is flat and never meets the axis, so the formula divides by zero.

d. $h_1 = 1 - \frac{-0.090141}{-3} = 0.969953$, $h_2 = 0.969953 - \frac{-0.000027}{-2.997292} = 0.969944$, $h_3 = 0.969944$. The depth is **0.9699 m**, about 97.0 cm.

e. The whole sphere holds $\frac{4}{3}\pi \approx 4.19$ m^3, so half-full ($h = 1$) is about 2.09 m^3. 2 m^3 is a little less than that, so the water is a little less than half-way up.

MARKING SCHEME

- ✓ a. The equation from $V = 2$.
- ✓ b. Three sign changes: roots in $(-1, 0)$, $(0, 1)$, $(2, 3)$.
- ✓ The depth is the root in $(0, 1)$, since $0 \leq h \leq 2$.
- ✓ c. $f'(0) = f'(2) = 0$: flat tangents.
- ✓ d. $h_1 = 0.969953$.
- ✓ $h_2 = h_3 = 0.969944$: 0.9699 m.
- ✓ e. Half the sphere is about 2.09 m^3 > 2 m^3.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	12	17	**29**

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