Unit 20 · Numerical Computation

ANSWER KEY · FOR TEACHERS

Full marks 33

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 3

1. root
2. transcendental
3. intermediate value theorem
4. iteration
5. tangent
6. tolerance

Each carries 0.5 mark.

2 True or false

U · 3

i. ✗ **False**: There may be an even number of roots: $x^2 - 1$ is positive at -2 and 2, but has the roots -1 and 1 between them.

ii. ✓ **True**

iii. ✗ **False**: Halving is bisection's. Close to a root Newton–Raphson's error is about squared each step, so the correct decimal places about double.

iv. ✓ **True**

v. ✗ **False**: It is at most $\frac{1}{2^{10}} = \frac{1}{1024} \approx 0.001$.

vi. ✗ **False**: It has a logarithm of x in it, so it is transcendental.

Each carries 0.5 mark.

3 Multiple choice

i. K · 1

c. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

The tangent at x_n meets the X-axis at $x_n - \frac{f(x_n)}{f'(x_n)}$. With + the step goes away from the root; the second has the fraction upside down.

ii. U · 1

a. $[1, 1.5]$

$f(1) = -1$, $f(1.5) = 3.375 - 1.5 - 1 = 0.875$, $f(2) = 5$. The sign changes between 1 and 1.5, so keep $[1, 1.5]$. $[1.25, 1.5]$ is after the second step.

iii. U · 1

c. 0.0176%

$E_a = |0.5672 - 0.5671| = 0.0001$, and $\frac{0.0001}{0.5672} \times 100\% \approx 0.0176\%$. 0.0001% puts a % sign on the absolute error itself; 0.01% forgets to divide by x_4 ($0.0001 \times 100\%$); 1.76% slips two decimal places.

iv. A · 1

a. 2.1

$f(2) = 8 - 4 - 5 = -1$, $f'(x) = 3x^2 - 2$, $f'(2) = 10$: $x_1 = 2 - \frac{-1}{10} = 2.1$. 1.9 adds the fraction instead of subtracting it; 2.083 leaves the -2 out of f'; 2.5 is bisection's first midpoint of $[2, 3]$.

v. A · 1

c. 9

$\frac{4-1}{2^n} < 0.01$ means $2^n > 300$: $2^8 = 256$ is too few, $2^9 = 512$ is enough. 7 forgets that the interval is 3 wide ($2^7 > 100$); 300 is 2^n, not n.

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vi. **a.** +, −, − HA · 1

After 0.5 the next midpoint is 0.25, so the root was kept in [0, 0.5]: $f(0.5)$ has the sign of $f(1)$, +. Then 0.375 means [0.25, 0.5]: $f(0.25)$ has the sign of $f(0)$, −. Then 0.4375 means [0.375, 0.5]: $f(0.375) < 0$ too.

4 Short questions

a. ANSWER A · 5

$f(x) = e^x - 3x$: $f(0) = 1 > 0$, $f(1) = 2.7183 - 3 = -0.2817 < 0$. f is continuous, so there is a root in (0, 1). Here $f(c) > 0$ moves a up to c, $f(c) < 0$ moves b down.

n	a	b	c	$f(c)$
1	0	1	0.5	0.1487
2	0.5	1	0.75	−0.1330
3	0.5	0.75	0.625	−0.0068
4	0.5	0.625	0.5625	0.0676
5	0.5625	0.625	0.59375	0.0295
6	0.59375	0.625	0.609375	0.0112
7	0.609375	0.625	0.6171875	0.0021
8	0.6171875	0.625	0.62109375	−0.0023

The root is in [0.6171875, 0.62109375]; both ends round to 0.62. The root is **0.62** correct to two decimal places.

MARKING SCHEME

- ✓ $f(0) > 0$ and $f(1) < 0$, so a root in (0, 1).
- ✓ Steps 1 to 3 right (midpoints 0.5, 0.75, 0.625 with the signs of f).
- ✓ Steps 4 to 6 right.
- ✓ Steps 7 and 8 right, ending in [0.6171875, 0.62109375].
- ✓ 0.62, since both ends round to it.

b. ANSWER U · 5

a. The tangent has slope $f'(x_n)$, so its equation is $y - f(x_n) = f'(x_n)(x - x_n)$. It meets the X-axis where $y = 0$: $-f(x_n) = f'(x_n)(x_{n+1} - x_n)$. If $f'(x_n) \neq 0$, dividing gives $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. It can't be used when $f'(x_n) = 0$: the tangent is parallel to the X-axis and never meets it.

b. The root is always inside the interval, and the midpoint of an interval is at most half its width from any point in it. The first interval is $b - a$ wide and each step halves it, so the nth midpoint is within $\dfrac{1}{2} \cdot \dfrac{b-a}{2^{n-1}} = \dfrac{b-a}{2^n}$ of the root. $\dfrac{1}{2^n} < 0.001$ needs $2^n > 1000$: **10 steps** ($2^{10} = 1024$).

MARKING SCHEME

- ✓ a. The tangent $y - f(x_n) = f'(x_n)(x - x_n)$.
- ✓ $y = 0$ put in and solved for x_{n+1}.
- ✓ It fails when $f'(x_n) = 0$ (a flat tangent).
- ✓ b. The root stays in the interval, which halves each step, and the midpoint is within half the width.
- ✓ 10 steps, from $2^{10} = 1024 > 1000$.

5 Long question

ANSWER

a. $f(x) = xe^x - 2$: $f(0) = -2 < 0$, $f(1) = e - 2 = 0.7183 > 0$, and f is continuous: a root in (0, 1).

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b. A · 8

n	a	b	c	$f(c)$
1	0	1	0.5	−1.1756
2	0.5	1	0.75	−0.4122
3	0.75	1	0.875	0.0990
4	0.75	0.875	0.8125	−0.1690

The root is in **[0.8125, 0.875]**, and $c_4 = 0.8125$ is within $\frac{1}{2^4} = 0.0625$ of it.

c. $f'(x) = e^x(x+1)$, so $x_{n+1} = x_n - \frac{x_n e^{x_n} - 2}{e^{x_n}(x_n + 1)}$.

$x_1 = 1 - \frac{0.718282}{5.436564} = 0.867879$, $x_2 = 0.867879 - \frac{0.067161}{4.449014} = 0.852783$,

$x_3 = 0.852783 - \frac{0.000771}{4.346939} = 0.852606$, $x_4 = 0.852606$.

The root is **0.8526**.

d. From an interval of width 0.0625 ([0.8125, 0.875]), $\frac{0.0625}{2^n} < 0.0005$ needs $2^n > 125$: **7 more steps** ($2^7 = 128$).

MARKING SCHEME

- ✓ a. $f(0) < 0$, $f(1) > 0$: a root in (0, 1).
- ✓ b. Midpoints 0.5, 0.75 with negative f.
- ✓ Midpoints 0.875 (positive) and 0.8125 (negative).
- ✓ The root in [0.8125, 0.875], within 0.0625 of c_4.
- ✓ c. $f'(x) = e^x(x+1)$.
- ✓ $x_1 = 0.867879$, $x_2 = 0.852783$.
- ✓ $x_3 = x_4 = 0.852606$: 0.8526.
- ✓ d. $2^n > 125$: 7 more steps.

6 Think

ANSWER HA · 3

$f(x) = x^3 + x - 3$ has $f'(x) = 3x^2 + 1 > 0$ for every x, so f is increasing everywhere (Unit 18): its graph crosses the X-axis at most once. $f(1) = -1 < 0$ and $f(2) = 7 > 0$, and f is continuous, so it does cross, between 1 and 2. So there is exactly one real root, in (1, 2).

MARKING SCHEME

- ✓ $f'(x) = 3x^2 + 1 > 0$, so f is increasing: at most one root.
- ✓ $f(1) < 0 < f(2)$ (a sign change), so at least one.
- ✓ Exactly one, in (1, 2).

7 Activity: Watch the digits double

(not marked)

WHAT STUDENTS SHOULD FIND

$x_1 = 1.5$, $x_2 = 1.416666667$, $x_3 = 1.414215686$, $x_4 = 1.414213562$, with 0, 2, 5 and 11 correct decimal places (x_4 is right in every place a 10-digit calculator shows). The correct places about double each step, as 20.3 says; from x_4 on the calculator shows the same number every time.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	4	10	15	4	**33**

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