Unit 19 · Anti-derivatives

HARDER PRACTICE QUESTIONS

Full marks 33

Write your answers on separate paper. Show your working.

1 Three answers to one integral [5]

Three students integrated $\int \tan x \sec^2 x\,dx$. Apekshya wrote $\frac{1}{2}\tan^2 x + C$, Saugat wrote $\frac{1}{2}\sec^2 x + C$, and Karma, seeing that both looked right, wrote $\frac{1}{2}\tan^2 x + \frac{1}{2}\sec^2 x + C$.

a. Show which substitution gives each of Apekshya's and Saugat's answers, and find the constant by which their anti-derivatives differ. [2]

b. Is Karma's answer right? If not, what is it the integral of? [1]

c. The curve whose slope is $\tan x \sec^2 x$ passes through $(0, 1)$. Find its equation from Apekshya's form and again from Saugat's, and show the two equations are the same curve. [2]

2 A lift that goes up and comes back [6]

A lift in a building starts from the ground floor. For the first 5 seconds its velocity, upward positive, is $v = t^2 - 6t + 8$ m/s. (The numbers are made up for the problem.)

a. During which seconds is the lift going down? [1]

b. How high above its start is it at $t = 5$? [2]

c. How far does it travel in the 5 seconds? [2]

d. At what time or times in $0 \le t \le 5$ is it highest? [1]

3 An integral that needs two methods [5]

a. Find $\int e^{\sqrt{x}}\,dx$. [3]

b. Hence evaluate $\int_0^1 e^{\sqrt{x}}\,dx$, and check that your answer is between the least and the greatest value $e^{\sqrt{x}}$ can take on $[0, 1]$. [2]

4 The area grows at the rate of the height [5]

f is continuous and $f(x) \ge 0$ for $x \ge 1$. The area under $y = f(x)$ from $x = 1$ to $x = t$ is $A(t) = t^3 - 3t + 2$ for every $t \ge 1$.

a. Find $f(x)$. [2]

b. Find the area under the curve from $x = 2$ to $x = 3$ in two ways. [1]

c. Explain why neither $t^3 - 3t$ nor $t^2 - 4t + 3$ could be such an area $A(t)$, for any $f \ge 0$. [2]

5 Trapping ln 2 between rectangles [6]

$\ln 2 = \int_1^2 \frac{dx}{x}$. Cut $[1, 2]$ into 4 strips of width 0.25.

a. For which choice of heights, the strips' left ends or their right ends, do the rectangles lie above the curve $y = \frac{1}{x}$? Why? [1]

b. Work out both sums, and so write ln 2 between two fractions. [3]

c. Show that, with n strips, the gap between the two sums is $\frac{1}{2n}$, and find the least n that traps ln 2 between two sums less than 0.01 apart. [2]

6 Halving the hump [6]

The parabola $y = 4x - x^2$ and the X-axis enclose a hump. A line $y = mx$, with $0 < m < 4$, through the origin cuts the hump into two pieces.

a. Show that the area between the parabola and the line is $\frac{(4-m)^3}{6}$. [3]

b. Find m, correct to three decimal places, so that the line cuts the hump into two equal areas. [2]

c. Prakriti guessed $m = 2$, half the parabola's slope at the origin. What fraction of the hump lies above that line? [1]