Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Three answers to one integral

ANSWER HA · 5

Tests: any two anti-derivatives differ by a constant, so different-looking answers can both be right; but adding two right answers is not right. The tempting answer is that Karma is right, "since both parts are".

a. $u = \tan x$, $du = \sec^2 x\,dx$: $\int u\,du = \frac{1}{2}\tan^2 x + C$. $u = \sec x$, $du = \sec x \tan x\,dx$: $\int \sec x \cdot \sec x \tan x\,dx = \int u\,du = \frac{1}{2}\sec^2 x + C$. Since $\sec^2 x = 1 + \tan^2 x$, $\frac{1}{2}\sec^2 x = \frac{1}{2}\tan^2 x + \frac{1}{2}$: they differ by the constant $\frac{1}{2}$.

b. **No.** Its derivative is $\tan x \sec^2 x + \tan x \sec^2 x = 2 \tan x \sec^2 x$: it is an anti-derivative of **twice** the integrand. (Half of it, $\frac{1}{4}\tan^2 x + \frac{1}{4}\sec^2 x$, would be right.)

c. Apekshya: $1 = 0 + C$, $C = 1$: $y = \frac{1}{2}\tan^2 x + 1$. Saugat: $1 = \frac{1}{2} + C$, $C = \frac{1}{2}$: $y = \frac{1}{2}\sec^2 x + \frac{1}{2} = \frac{1}{2}(1 + \tan^2 x) + \frac{1}{2} = \frac{1}{2}\tan^2 x + 1$. The same curve: the point fixes a different C for each form, and the difference in C makes up the $\frac{1}{2}$.

MARKING SCHEME

- ✓ Both substitutions, $u = \tan x$ and $u = \sec x$, shown.
- ✓ They differ by $\frac{1}{2}$ (from $\sec^2 x = 1 + \tan^2 x$).
- ✓ Karma is wrong: his answer's derivative is $2 \tan x \sec^2 x$.
- ✓ $C = 1$ in Apekshya's form and $C = \frac{1}{2}$ in Saugat's.
- ✓ Both give $y = \frac{1}{2}\tan^2 x + 1$.

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2 A lift that goes up and comes back

ANSWER A · 6

Tests: a definite integral of velocity is the displacement, a signed total; the distance needs the integral split where v changes sign. The tempting wrong answer is $\frac{20}{3}$ m for c, the integral straight across.

a. $v = (t-2)(t-4) < 0$ for $2 < t < 4$: it goes down between 2 s and 4 s.

b. $s = \int_0^5 (t^2 - 6t + 8)\,dt = \left[\ \frac{t^3}{3} - 3t^2 + 8t\ \right]_0^5 = \frac{125}{3} - 75 + 40 = \frac{20}{3} \approx 6.67$ m.

c. With $s(t) = \frac{t^3}{3} - 3t^2 + 8t$: $s(2) = \frac{20}{3}$, $s(4) = \frac{16}{3}$, $s(5) = \frac{20}{3}$. Up $\frac{20}{3}$, down $\frac{4}{3}$, up $\frac{4}{3}$: distance $\frac{20}{3} + \frac{4}{3} + \frac{4}{3} = \frac{28}{3} \approx 9.33$ m.

d. The highest point is where s is largest: at a turning point where v goes from + to −, or at an end. $s(2) = s(5) = \frac{20}{3}$: it is highest **twice**, at $t = 2$ s and again at $t = 5$ s. (At $t = 4$ it is lowest after the start, $\frac{16}{3}$ m.)

MARKING SCHEME

✓ a. Going down for $2 < t < 4$.

✓ b. $s(t) = \frac{t^3}{3} - 3t^2 + 8t$.

✓ $\frac{20}{3}$ m.

✓ c. Split at 2 and 4: up $\frac{20}{3}$, down $\frac{4}{3}$, up $\frac{4}{3}$.

✓ Distance $\frac{28}{3}$ m.

✓ d. At $t = 2$ s and $t = 5$ s, both at $\frac{20}{3}$ m.

3 An integral that needs two methods

ANSWER A · 5

Tests: choosing a substitution that turns the integrand into one for by parts. The tempting wrong answer is $e^{\sqrt{x}} + C$ (or $2\sqrt{x}\,e^{\sqrt{x}}$), treating $\sqrt{x}$ as if its derivative were 1.

a. Put $t = \sqrt{x}$, so $x = t^2$ and $dx = 2t\,dt$: $\int e^t \cdot 2t\,dt$. By parts, $u = t$: $2(te^t - e^t) + C = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$.

Check: $\frac{d}{dx}\left[2e^{\sqrt{x}}(\sqrt{x} - 1)\right] = 2e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}(\sqrt{x} - 1) + 2e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = e^{\sqrt{x}}$. ✓

b. $\left[\ 2e^{\sqrt{x}}(\sqrt{x} - 1)\ \right]_0^1 = 0 - 2(1)(-1) = 2$. On $[0, 1]$, $e^{\sqrt{x}}$ rises from 1 to $e \approx 2.72$, so the area over an interval of width 1 must be between 1×1 and $1 \times e$: 2 is. ✓

MARKING SCHEME

✓ $t = \sqrt{x}$, $dx = 2t\,dt$.

✓ $2\int te^t\,dt$ by parts with $u = t$.

✓ $2e^{\sqrt{x}}(\sqrt{x} - 1) + C$.

✓ $\int_0^1 = 2$.

✓ $1 < 2 < e$, from the least and greatest heights.

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4 The area grows at the rate of the height

ANSWER HA · 5

Tests: the fundamental theorem read backwards, $A'(t) = f(t)$, and what an area function must do: start at 0 and never decrease. The tempting slip in c is to check only one of the two conditions.

a. By the fundamental theorem $A'(x) = f(x)$: $f(x) = 3x^2 - 3$ (which is ≥ 0 for $x \geq 1$, as it must be).

b. $A(3) - A(2) = 20 - 4 = 16$; and $\int_2^3 (3x^2 - 3)\,dx = \left[\, x^3 - 3x \,\right]_2^3 = 18 - 2 = 16$. ✓

c. An area from 1 to t is 0 at $t = 1$ (no width), and it can't decrease as t grows, since $A'(t) = f(t) \geq 0$. $t^3 - 3t$ is -2 at $t = 1$, not 0. $t^2 - 4t + 3$ is 0 at $t = 1$, but its derivative $2t - 4$ is negative for $1 < t < 2$: it would need $f < 0$ there (it would make the area negative, -1 at $t = 2$).

MARKING SCHEME

✓ $A'(x) = f(x)$ used.
✓ $f(x) = 3x^2 - 3$.
✓ 16 both ways.
✓ $t^3 - 3t$ fails: $A(1)$ must be 0.
✓ $t^2 - 4t + 3$ fails: it decreases on $(1, 2)$, which needs $f < 0$.

5 Trapping ln 2 between rectangles

ANSWER A · 6

Tests: which rectangle sum is too big depends on whether the curve rises or falls; the gap between the sums telescopes. The tempting slip is to say the right ends overshoot, as they do for the rising x^2 in the unit.

a. The **left** ends. $\frac{1}{x}$ is decreasing (its derivative $-\frac{1}{x^2} < 0$), so on each strip the left end is the highest point, and the rectangle covers the curve.

b. Left: $0.25\left(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) = \frac{319}{420} \approx 0.7595$. Right: $0.25\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\right) = \frac{533}{840} \approx 0.6345$.

So $\frac{533}{840} < \ln 2 < \frac{319}{420}$, that is $0.6345 < \ln 2 < 0.7595$. ($\ln 2 = 0.6931$.)

c. The two sums share every height but the first and last: left minus right $= \Delta x[f(1) - f(2)] = \frac{1}{n}\left(1 - \frac{1}{2}\right) = \frac{1}{2n}$. For $\frac{1}{2n} < 0.01$, $n > 50$: **51 strips**. (With 4 strips the gap is $\frac{1}{8} = 0.125$, as in b.)

MARKING SCHEME

✓ a. Left ends, because $\frac{1}{x}$ decreases.
✓ b. Left sum ≈ 0.7595.
✓ Right sum ≈ 0.6345.
✓ $0.6345 < \ln 2 < 0.7595$, with the inequalities the right way round.
✓ c. Only the end heights differ: gap $\frac{1}{n}\left(1 - \frac{1}{2}\right) = \frac{1}{2n}$.
✓ $n = 51$.

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6 Halving the hump

ANSWER HA · 6

Tests: an area between curves with a parameter in the limits; setting an area equal to a target. The tempting wrong answer is $m = 2$, halving the slope instead of the area.

a. They meet where $4x - x^2 = mx$: $x(4 - m - x) = 0$, $x = 0$ or $x = 4 - m$. Between them the parabola is on top (at $x = \dfrac{4-m}{2}$ its height exceeds the line's by $\dfrac{(4-m)^2}{4} > 0$). Write $k = 4 - m$:

$\int_0^k (kx - x^2)\,dx = \left[\ \dfrac{kx^2}{2} - \dfrac{x^3}{3}\ \right]_0^k = \dfrac{k^3}{2} - \dfrac{k^3}{3} = \dfrac{k^3}{6} = \dfrac{(4-m)^3}{6}$.

b. The hump: $\int_0^4 (4x - x^2)\,dx = 32 - \dfrac{64}{3} = \dfrac{32}{3}$ (also $m = 0$ in a). Half of it: $\dfrac{(4-m)^3}{6} = \dfrac{16}{3}$, so $(4-m)^3 = 32$, $4 - m = \sqrt[3]{32} \approx 3.175$, and $m \approx 0.825$.

c. $m = 2$: $\dfrac{2^3}{6} = \dfrac{4}{3}$, and $\dfrac{4/3}{32/3} = \dfrac{1}{8}$: only an eighth of the hump is above the line. Since the area goes as the cube of $4 - m$, halving $4 - m$ divides the area by 8.

MARKING SCHEME

- ✓ a. Meeting points $x = 0$ and $x = 4 - m$, parabola on top.
- ✓ $\int_0^{4-m} [(4-m)x - x^2]\,dx$ set up.
- ✓ Worked out to $\dfrac{(4-m)^3}{6}$.
- ✓ b. The hump is $\dfrac{32}{3}$, so $(4-m)^3 = 32$.
- ✓ $m \approx 0.825$.
- ✓ c. $\dfrac{1}{8}$ of the hump.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	17	16	**33**

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