Unit 19 · Anti-derivatives

ANSWER KEY · FOR TEACHERS

Full marks 33

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 3

1. anti-derivative
2. integrand
3. constant of integration
4. definite integral
5. lower limit
6. substitution

Each carries 0.5 mark.

2 True or false

U · 3

i. ✗ **False**: There is no product rule like this: $\int x \cdot x\,dx = \frac{x^3}{3} + C$, but $\int x\,dx \cdot \int x\,dx = \frac{x^4}{4}$.
A product is multiplied out, or done by substitution or by parts.

ii. ✗ **False**: $\int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$; only $\frac{1}{x}$ integrates to a logarithm.

iii. ✓ **True**

iv. ✓ **True**

v. ✗ **False**: The part from -1 to 0 is below the axis and counts negative: the area is $\frac{1}{2} + 2 = \frac{5}{2}$.

vi. ✗ **False**: The limits change too: $x = 0 \to u = 1$, $x = 1 \to u = 2$, so it is $\frac{1}{2}\int_1^2 u^3\,du = \frac{15}{8}$.

Each carries 0.5 mark.

3 Multiple choice

i. **b.** $\tan x + C$ — K · 1

$\frac{d}{dx}\tan x = \sec^2 x$. The last is the derivative of $\sec^2 x$, not its integral.

ii. **a.** $\frac{1}{3}\sin(3x+1) + C$ — U · 1

A linear inside: integrate as if it were x, then divide by 3. $\sin(3x+1)$ has derivative $3\cos(3x+1)$, three times too much; the minus sign belongs to $\int \sin x\,dx$, not $\int \cos x\,dx$.

iii. **c.** 4 — U · 1

$x^3 - 3x$ is odd, so its part is 0; the constant 1 gives $\int_{-2}^{2} 1\,dx = 4$. 0 forgets the constant; 2 takes only $\int_0^2 1\,dx$; 8 doubles the 4 again.

iv. **a.** $x\tan x - \ln|\sec x| + C$ — A · 1

By parts, $u = x$, $v = \sec^2 x$: $x\tan x - \int \tan x\,dx = x\tan x - \ln|\sec x| + C$. Check: the derivative is $\tan x + x\sec^2 x - \tan x$. ✓ The second gets the sign of the last integral wrong (its derivative has an extra $2\tan x$); the third stops after the first term; the last integrates each factor on its own.

v. **b.** 240 square units — A · 1

$4x^3 \geq 0$ on $[2, 4]$: $\int_2^4 4x^3\,dx = \left[\, x^4 \,\right]_2^4 = 256 - 16 = 240$. 256 starts from 0; 272 adds $F(2)$ instead of taking it away; 48 is the slope $12x^2$ at $x = 2$.

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vi. **c.** 11 HA · 1

$\int_2^5 f = 7 - 3 = 4$ (ranges add), so $\int_2^5 (2f+1)\,dx = 2\times 4 + (5-2) = 11$. 8 forgets $\int_2^5 1\,dx = 3$; 9 adds only 1 for it; 17 uses 7 for $\int_2^5 f$.

4 Short questions

a. A · 5

ANSWER

a. The top is the derivative of the bottom: $u = x^2 + 3x + 5$, $du = (2x+3)\,dx$, so $\int \frac{du}{u} = \ln(x^2+3x+5) + C$ (no bars: $x^2 + 3x + 5 > 0$ for every x, since $9 - 20 < 0$).

b. $u = x - 1$, so $x = u + 1$ and $dx = du$:
$\int (u+1)u^{1/2}\,du = \int (u^{3/2} + u^{1/2})\,du = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C.$

MARKING SCHEME

✓ a. $u = x^2 + 3x + 5$, $du = (2x+3)\,dx$.
✓ $\ln(x^2+3x+5) + C$.
✓ b. $u = x - 1$ with $x = u + 1$, $dx = du$.
✓ $\int (u^{3/2} + u^{1/2})\,du$.
✓ $\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$.

b. U · 5

ANSWER

a. For functions u and w of x, the product rule gives $(uw)' = uw' + wu'$. Integrating both sides, $uw = \int uw'\,dx + \int wu'\,dx$, so $\int uw'\,dx = uw - \int wu'\,dx$. Writing v for w', so that $w = \int v\,dx$:
$\int u\,v\,dx = u\int v\,dx - \int \left(\frac{du}{dx}\int v\,dx \right) dx.$

b. $u = x$: differentiating makes it 1, and the new integral $\int 1 \cdot (-e^{-x})\,dx$ is easy:
$\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C$. With $u = e^{-x}$ and $v = x$, the new integral is
$\int (-e^{-x})\frac{x^2}{2}\,dx$: the power of x goes up, so it is harder than the one we started with.

MARKING SCHEME

✓ a. The product rule $(uw)' = uw' + wu'$ integrated.
✓ Rearranged to $\int uw'\,dx = uw - \int wu'\,dx$.
✓ $v = w'$, $w = \int v\,dx$ put in to give the formula.
✓ b. $u = x$, since the other choice raises the power of x in the new integral.
✓ $-e^{-x}(x+1) + C$.

5 Long question

ANSWER

a. $x^2 - 4x + 3 = x - 1$: $x^2 - 5x + 4 = 0$, $(x-1)(x-4) = 0$. They meet at (1, 0) and (4, 3). The parabola opens upward, crosses the X-axis at 1 and 3, and has its lowest point at (2, −1).

b. At $x = 2$ the line is at 1 and the curve at −1: the line is on top.

$\int_1^4 [(x-1) - (x^2 - 4x + 3)]\,dx = \int_1^4 (-x^2 + 5x - 4)\,dx = \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x \right]_1^4 = \frac{8}{3} - \left(-\frac{11}{6}\right) = \frac{9}{2}$
square units.

c. $\left[\frac{x^3}{3} - 2x^2 + 3x \right]_0^4 = \frac{64}{3} - 32 + 12 = \frac{4}{3}$.

Split where the curve crosses the axis: $\int_0^1 = \frac{4}{3}$, $\int_1^3 = -\frac{4}{3}$ (below the axis), $\int_3^4 = \frac{4}{3}$. Area
$= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$ square units.
The integral counts the part below the axis as negative, so it cancels one of the parts above; the area adds the sizes of all three.

MARKING SCHEME

✓ a. $x^2 - 5x + 4 = 0$, meeting at (1, 0) and (4, 3).
✓ A sketch with the parabola's vertex (2, −1) and the line through both points.
✓ b. The line on top on [1, 4], integrand $-x^2 + 5x - 4$.
✓ The anti-derivative $-\frac{x^3}{3} + \frac{5x^2}{2} - 4x$.

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✓ $\frac{9}{2}$ square units. **A · 8**

✓ c. $\int_0^4 = \frac{4}{3}$.

✓ The pieces $\frac{4}{3}, -\frac{4}{3}, \frac{4}{3}$ (split at 1 and 3): area 4.

✓ The part below the axis counts negative in the integral.

6 Think

ANSWER **HA · 3**

$\frac{1}{x^2} > 0$ everywhere it is defined, so every rectangle $f(x_k)\,\Delta x$ in the sum is positive, and the integral of a positive function can't be negative. The step that fails is $\int_a^b f(x)\,dx = F(b) - F(a)$: it needs f continuous on all of $[a, b]$, and $\frac{1}{x^2}$ is not defined at $x = 0$, inside $[-1, 1]$, where $-\frac{1}{x}$ is not an anti-derivative across the gap. (Near 0 the curve shoots up, and the area is not a finite number at all.)

MARKING SCHEME

✓ The integrand is positive, so the sum of rectangles (the area) can't be negative.

✓ $F(b) - F(a)$ needs f continuous on the whole interval.

✓ $\frac{1}{x^2}$ is not defined (not continuous) at $x = 0$, inside $[-1, 1]$.

7 Activity: An area by strips

(not marked)

WHAT STUDENTS SHOULD FIND

The heights are 0.9375, 2.4375, 3.4375, 3.9375, 3.9375, 3.4375, 2.4375, 0.9375 (symmetric about $x = 2$); the strips add to 10.75. The integral is $\left[\, 2x^2 - \frac{x^3}{3} \,\right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3} \approx 10.67$. The strips are less than 1% too much; narrower strips would come closer, as in 19.6.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	4	10	15	4	**33**

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