Unit 18 · Derivatives

HARDER PRACTICE QUESTIONS

Full marks 31

Write your answers on separate paper. Show your working.

1 A smooth join [5]

A function is given in two pieces:

$f(x)$	when
x^2	$x \le 1$
$ax + b$	$x > 1$

a. Find a and b so that f is differentiable at $x = 1$. [3]

b. With these values, does $f''(1)$ exist? Explain what that means for the graph. [1]

c. Show that $g(x) = x|x|$ is differentiable at $x = 0$, though $|x|$ is not. [1]

2 One question, two answers [5]

$f(x) = x^4 + kx^3$, where k is a constant.

a. Show that $f'(0) = 0$ and $f''(0) = 0$, whatever k is. [1]

b. For $k = 0$, and then for $k \neq 0$, decide whether $x = 0$ gives a local maximum, a local minimum, a point of inflection, or none of these. Give your reasons. [3]

c. For $k \neq 0$, find the other stationary point and say what kind it is. [1]

3 A box for a shelf [6]

Equal squares of side x cm are cut from the corners of a square sheet of tin 60 cm wide, and the sides are folded up to make an open box.

a. Show that the box holds $V = x(60 - 2x)^2$ cm^3, and find the x that makes V as large as possible, with that volume. [3]

b. The box must fit on a shelf, so it can be at most 8 cm high. Find the largest volume now, and explain why it is not at a stationary point. [3]

4 Which way does the arch bend? [5]

A stone stuck in the tyre of a rolling wheel of radius 1 traces the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$, one arch for $0 < \theta < 2\pi$.

a. Show that $\dfrac{dy}{dx} = \cot\dfrac{\theta}{2}$. [1]

b. Find $\dfrac{d^2y}{dx^2}$ in terms of θ. [2]

c. Nishant divides $\dfrac{d^2y}{d\theta^2}$ by $\dfrac{d^2x}{d\theta^2}$, gets $\cot\theta$, and says the arch bends upward for $0 < \theta < \dfrac{\pi}{2}$. Decide which way the arch really bends, and say what is wrong with his method. [2]

5 Squeezing sin x [5]

a. Show that $\sin x < x$ for every $x > 0$. [1]

b. Show that $x - \dfrac{x^3}{6} < \sin x$ for every $x > 0$. [3]

c. Without a calculator, use a. and b. to say how far $\sin 0.1$ can be from 0.1 at most. [1]

6 How many roots? [5]

a. Show that $e^x = 3x$ has exactly two real roots. (Take $\ln 3 \approx 1.10$.) [3]

b. For which positive number k does $e^x = kx$ have exactly one root? [2]