Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 A smooth join

ANSWER HA · 5

Tests: differentiable means continuous **and** the two one-sided slopes equal; the tempting wrong answer stops at continuity ($a + b = 1$), which leaves a free and allows a corner.

a. Continuity at 1: LHL $= 1 = f(1)$ and RHL $= a + b$, so $a + b = 1$. Equal slopes: from the left $\frac{d}{dx}(x^2) = 2x = 2$ at 1; from the right the slope is a. So $a = 2$, and then $\mathbf{a = 2}$, $\mathbf{b = -1}$: the line $y = 2x - 1$ is the parabola's tangent at $(1, 1)$. (Without continuity, equal slopes alone wouldn't do: $a = 2$, $b = 0$ has a jump.)

b. **No.** $f'(x) = 2x$ for $x < 1$ and 2 for $x > 1$, so f' is continuous at 1, but its slope is 2 from the left and 0 from the right. The graph joins without a corner, but it stops bending there: concave up on the left, straight on the right.

c. $\frac{g(0+h) - g(0)}{h} = \frac{h|h|}{h} = |h| \to 0$ from both sides, so $g'(0) = 0$. For $|x|$ the same fraction is $\frac{|h|}{h} = \pm 1$: the two sides differ. The extra factor x flattens the corner.

MARKING SCHEME

- ✓ a. Continuity: $a + b = 1$.
- ✓ a. Equal slopes: $a = 2$.
- ✓ a. $b = -1$.
- ✓ b. No: f'' is 2 on the left and 0 on the right.
- ✓ c. $\frac{h|h|}{h} = |h| \to 0$ from both sides, so $g'(0) = 0$.

2 One question, two answers

ANSWER HA · 5

Tests: when $f'(c) = f''(c) = 0$ the second derivative test says nothing, and the answer has to come from the signs of f' (for an extreme) and of f'' (for an inflection) on each side. The tempting wrong answer is one verdict for every k, since the values at 0 are the same for all of them.

a. $f'(x) = 4x^3 + 3kx^2 = x^2(4x + 3k)$ and $f''(x) = 12x^2 + 6kx = 6x(2x + k)$: both are 0 at $x = 0$ for every k.

b. $\mathbf{k = 0}$**:** $f = x^4$, $f' = 4x^3$ changes from $-$ to $+$ at 0: **a local minimum**. $f'' = 12x^2 \geq 0$ on both sides: concave up on both sides, **not an inflection**. $\mathbf{k \neq 0}$**:** near 0, $x^2 > 0$ and $4x + 3k$ has the sign of k, so f' has the same sign on both sides: **no extreme**. Near 0, $2x + k$ has the sign of k, so f'' has the sign of kx, which changes at 0: **a point of inflection with a flat tangent**.

c. $f' = 0$ also at $x = -\frac{3k}{4}$, where $f'' = 6\left(-\frac{3k}{4}\right)\left(-\frac{k}{2}\right) = \frac{9k^2}{4} > 0$: **a local minimum**, $f = -\frac{27k^4}{256}$.

MARKING SCHEME

- ✓ a. $f' = x^2(4x + 3k)$ and $f'' = 6x(2x + k)$, both 0 at 0.
- ✓ b. $k = 0$: f' changes from $-$ to $+$, a minimum.
- ✓ b. $k = 0$: f'' doesn't change sign, no inflection.
- ✓ b. $k \neq 0$: f' keeps its sign (no extreme) and f'' changes sign (an inflection).
- ✓ c. $x = -\frac{3k}{4}$, $f'' = \frac{9k^2}{4} > 0$: a local minimum.

3 A box for a shelf

ANSWER

Tests: a best value at an end of the allowed interval, when the stationary point is outside it. The tempting wrong answer in b. is $x = 10$ again, or to say there is no answer because $V' \neq 0$ for $0 < x \leq 8$.

a. The base is $(60 - 2x)$ cm square and the height x cm, so $V = x(60 - 2x)^2$, with $0 < x < 30$.

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking

$V'(x) = (60-2x)^2 - 4x(60-2x) = (60-2x)(60-6x) = 12(x-30)(x-10)$. In $(0, 30)$, $V' = 0$ only at $x = 10$; $V' > 0$ before it and < 0 after: a maximum. **$x = 10$ cm, $V = 10 \times 40^2 = 16000$ cm^3.** A · 6

b. Now $0 < x \leq 8$. On $(0, 10)$, $V' > 0$ (both factors $x - 30$ and $x - 10$ are negative), so V is increasing all the way to 8: the largest value is at the end, **$x = 8$: $V = 8 \times 44^2 = 15488$ cm^3**. There is no stationary point in $(0, 8]$; the best is where the shelf stops the box from growing, as the highest point of a climbing road is where the road ends.

MARKING SCHEME

- ✓ a. $V = x(60-2x)^2$ from the base and height.
- ✓ a. $V' = 12(x-30)(x-10)$, zero at $x = 10$ in the interval.
- ✓ a. A maximum at $x = 10$: 16,000 cm^3.
- ✓ b. $V' > 0$ on $(0, 8]$, so V is increasing there.
- ✓ b. The largest value is at the end $x = 8$.
- ✓ b. 15,488 cm^3.

4 Which way does the arch bend?

ANSWER A · 5

Tests: the second derivative of a curve given by a parameter, and judging an answer by what the curve looks like. The tempting wrong answer is Nishant's ratio, which copies the first-derivative rule.

a. $\dfrac{dx}{d\theta} = 1 - \cos\theta = 2\sin^2\dfrac{\theta}{2}$ and $\dfrac{dy}{d\theta} = \sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$, so $\dfrac{dy}{dx} = \cot\dfrac{\theta}{2}$.

b. $$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right) \div \frac{dx}{d\theta} = -\frac{1}{2\sin^2\dfrac{\theta}{2}} \div 2\sin^2\frac{\theta}{2} = -\frac{\mathbf{1}}{\mathbf{4\sin^4\dfrac{\theta}{2}}}.$$

c. By b., $\dfrac{d^2y}{dx^2} < 0$ for every θ in $(0, 2\pi)$: **the arch is concave down all along**, a cap, as it must be: it rises from the road to its top $y = 2$ at $\theta = \pi$ and comes down again. Nishant's $\cot\theta > 0$ for $0 < \theta < \dfrac{\pi}{2}$ is wrong: $\dfrac{d^2y}{dx^2}$ is the rate of change of $\dfrac{dy}{dx}$ with respect to x, so it is $\dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right) \div \dfrac{dx}{d\theta}$, not a ratio of second derivatives.

MARKING SCHEME

- ✓ a. $\dfrac{dy}{dx} = \dfrac{\sin\theta}{1-\cos\theta} = \cot\dfrac{\theta}{2}$.
- ✓ b. $\dfrac{d}{d\theta}\left(\cot\dfrac{\theta}{2}\right) = -\dfrac{1}{2\sin^2(\theta/2)}$, divided by $\dfrac{dx}{d\theta}$.
- ✓ b. $-\dfrac{1}{4\sin^4(\theta/2)}$.
- ✓ c. Negative everywhere: concave down along the whole arch.
- ✓ c. Nishant's ratio is not $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$.

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking

5 Squeezing sin x

ANSWER HA · 5

Tests: proving an inequality by monotonicity, used more than once in a chain, where the derivative of the difference has no sign that can be seen straight away. The tempting wrong step is to stop when $g' = \cos x - 1 + \frac{x^2}{2}$ has no obvious sign, or to check a few values instead.

a. Let $h(x) = x - \sin x$. $h'(x) = 1 - \cos x \geq 0$, and is 0 only at single points, so h is increasing on $[0, \infty)$. For $x > 0$, $h(x) > h(0) = 0$: $\sin x < x$.

b. Let $g(x) = \sin x - x + \frac{x^3}{6}$, so $g(0) = 0$. Then $g'(x) = \cos x - 1 + \frac{x^2}{2}$, with $g'(0) = 0$, and $g''(x) = x - \sin x > 0$ for $x > 0$ (by a.). So g' is increasing on $[0, \infty)$: $g'(x) > g'(0) = 0$ for $x > 0$. So g is increasing too: $g(x) > g(0) = 0$, that is $\sin x > x - \frac{x^3}{6}$.

c. $0.1 - \frac{0.001}{6} < \sin 0.1 < 0.1$, so $\sin 0.1$ is less than 0.1 by at most $\frac{0.001}{6} \approx 0.00017$: between 0.09983 and 0.1. (A calculator gives 0.0998334, close to the lower bound.)

MARKING SCHEME

- ✓ a. $h = x - \sin x$, $h' = 1 - \cos x \geq 0$, $h(0) = 0$: $\sin x < x$.
- ✓ b. $g = \sin x - x + \frac{x^3}{6}$ with $g(0) = g'(0) = 0$.
- ✓ b. $g'' = x - \sin x > 0$ (from a.), so $g' > 0$.
- ✓ b. So g increasing, $g > 0$.
- ✓ c. Less than 0.1 by at most $\frac{0.001}{6} \approx 0.00017$.

6 How many roots?

ANSWER HA · 5

Tests: counting roots from where a function rises and falls and its least value, not by trying numbers. The tempting wrong answers: "the minimum is below 0, so there are no roots", or finding two roots by trial and not showing there are no more.

a. Let $g(x) = e^x - 3x$. $g'(x) = e^x - 3$: negative for $x < \ln 3$, positive after, so g decreases, then increases, and its least value is $g(\ln 3) = 3 - 3\ln 3 = 3(1 - \ln 3) < 0$, since $\ln 3 > 1$. On each side g also takes positive values: $g(0) = 1 > 0$, and $g(2) = e^2 - 6 \approx 1.39 > 0$. So g crosses 0 once while falling, between 0 and ln 3, and once while rising, between ln 3 and 2; on each piece it is monotonic, so it can't cross twice. **Exactly two roots** (about 0.62 and 1.51).

b. For $k > 0$ a root must have $x > 0$, since $e^x > 0$. Exactly one root when the line $y = kx$ just touches $y = e^x$: there the values and the slopes agree, $e^c = kc$ and $e^c = k$. So $kc = k$, $c = 1$, and $k = e$. (For $0 < k < e$ there is no root, for $k > e$ two, as in a. with $k = 3$.)

MARKING SCHEME

- ✓ a. $g' = e^x - 3$: decreasing, then increasing, turning at ln 3.
- ✓ a. The least value $3(1 - \ln 3) < 0$.
- ✓ a. $g > 0$ on both sides ($g(0) = 1$, $g(2) > 0$) and monotonic on each piece: exactly two roots.
- ✓ b. The line touches the curve: $e^c = kc$ and $e^c = k$.
- ✓ b. $c = 1$, $k = e$ (or, from a.'s route, the least value of $e^x - kx$, $k - k\ln k$, equal to 0).

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	11	20	**31**

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking