Unit 18 · Derivatives

ANSWER KEY · FOR TEACHERS

Full marks 33

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 3

1. secant
2. tangent
3. stationary point
4. local maximum
5. concave upward
6. point of inflection

Each carries 0.5 mark.

2 True or false

U · 3

i. ✗ **False**: Differentiable means continuous, not the other way round: $|x|$ is continuous at 0 but has a corner there, and no derivative.

ii. ✓ **True**

iii. ✗ **False**: $(uv)' = uv' + vu'$: $(x^2 \cdot x^3)' = 5x^4$, not $2x \cdot 3x^2$.

iv. ✗ **False**: An inflection needs f'' to change sign: x^4 has $f''(0) = 0$ and is concave up on both sides.

v. ✓ **True**

vi. ✗ **False**: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \div \dfrac{dx}{dt}$.

Each carries 0.5 mark.

3 Multiple choice

i. **b.** $-\operatorname{cosec} x \cot x$ — K · 1

$\operatorname{cosec} x = \dfrac{1}{\sin x}$, and the quotient rule gives $-\dfrac{\cos x}{\sin^2 x} = -\operatorname{cosec} x \cot x$. $-\operatorname{cosec}^2 x$ is the derivative of $\cot x$, and $\sec x \tan x$ of $\sec x$.

ii. **b.** $\dfrac{x}{\sqrt{1+x^2}}$ — U · 1

General power rule: $\dfrac{1}{2}(1+x^2)^{-1/2} \cdot 2x$. The first leaves out the inside's derivative $2x$; the third differentiates the inside and keeps the root; the fourth drops the root from the bottom.

iii. **a.** $\dfrac{x}{y}$ — U · 1

$2x - 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = \dfrac{x}{y}$. $-\dfrac{x}{y}$ is the circle's answer, from $x^2 + y^2 = r^2$: the minus sign on y^2 changes it.

iv. **c.** $81e^{3x}$ — U · 1

Each differentiation brings out one more 3 (the chain rule), so the fourth derivative is $3^4 e^{3x} = 81e^{3x}$. $12e^{3x}$ adds the four 3s instead of multiplying them; the last loses the $3x$ in the power.

v. **b.** $\dfrac{1}{e}$ — A · 1

$f'(x) = e^{-x}(1-x)$: + before $x = 1$, − after it, so the greatest value is $f(1) = e^{-1} = \dfrac{1}{e}$. 1 is where it happens, not the value.

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vi. **b.** $-3 \le a \le 3$ HA · 1

$f'(x) = 3x^2 + 2ax + 3$ must never be negative: a quadratic that opens upward, so its discriminant $4a^2 - 36$ must be ≤ 0, $a^2 \le 9$. (At $a = \pm 3$, $f' = 3(x \pm 1)^2$ is 0 at one point only, which doesn't stop f increasing.) $a = 0$ is one such value, not the only one.

4 Short questions

a. ANSWER U · 5

a. $$\frac{\sin(3x+3h) - \sin 3x}{h} = \frac{2\cos\left(3x + \frac{3h}{2}\right)\sin\frac{3h}{2}}{h} = 3\cos\left(3x + \frac{3h}{2}\right) \cdot \frac{\sin\frac{3h}{2}}{\frac{3h}{2}} \to \mathbf{3\cos 3x}.$$

b. Put $x = \tan\theta$, $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then $\frac{1+\tan\theta}{1-\tan\theta} = \tan\left(\frac{\pi}{4} + \theta\right)$, and $\frac{\pi}{4} + \theta$ is in $\left(0, \frac{\pi}{2}\right)$, so $y = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \tan^{-1} x$. $\frac{dy}{dx} = \mathbf{\frac{1}{1+x^2}}$.

MARKING SCHEME

- ✓ a. $\sin C - \sin D$ used on $\sin(3x+3h) - \sin 3x$.
- ✓ a. $3\cos 3x$, with $\frac{\sin(3h/2)}{3h/2} \to 1$.
- ✓ b. $x = \tan\theta$ and $\frac{1+\tan\theta}{1-\tan\theta} = \tan\left(\frac{\pi}{4} + \theta\right)$.
- ✓ b. $y = \frac{\pi}{4} + \tan^{-1} x$ (the angle in the principal range).
- ✓ b. $\frac{1}{1+x^2}$.

b. ANSWER A · 5

a. $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = 3t^2 - 3$, so $\frac{dy}{dx} = \mathbf{\frac{3(t^2-1)}{2}}$.

b. $\frac{dy}{dx} = 0$ at $t = \pm 1$: $t = 1$ gives $(3, -2)$, $t = -1$ gives $(-1, 2)$.

c. $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{3(t^2-1)}{2}\right) \div \frac{dx}{dt} = 3t \div 2 = \mathbf{\frac{3t}{2}}$. At $t = 1$ it is $\frac{3}{2} > 0$: **concave up**, so $(3, -2)$ is the bottom of a dip.

MARKING SCHEME

- ✓ a. $\frac{dy}{dx} = \frac{3(t^2-1)}{2}$.
- ✓ b. $t = \pm 1$.
- ✓ b. $(3, -2)$ and $(-1, 2)$.
- ✓ c. $\frac{d^2y}{dx^2} = \frac{3t}{2}$ (differentiated with respect to t, then divided by $\frac{dx}{dt}$).
- ✓ c. $\frac{3}{2} > 0$: concave up.

5 Long question

ANSWER

a. $f'(x) = 4x^3 - 16x = 4x(x-2)(x+2)$, zero at -2, 0 and 2. Its sign: − on $(-\infty, -2)$, + on $(-2, 0)$, − on $(0, 2)$, + on $(2, \infty)$. **Increasing on $(-2, 0)$ and $(2, \infty)$; decreasing on $(-\infty, -2)$ and $(0, 2)$.**

b. $f''(x) = 12x^2 - 16$. At $x = 0$: $f''(0) = -16 < 0$, a local maximum, $f(0) = \mathbf{3}$. At $x = \pm 2$: $f''(\pm 2) = 32 > 0$, local minima, $f(\pm 2) = 16 - 32 + 3 = \mathbf{-13}$.

c. $f''(x) = 0$ at $x^2 = \frac{4}{3}$, $x = \pm\frac{2}{\sqrt{3}}$; f'' is + outside them and − between. **Concave up on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$, concave down on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.** The sign changes at both, and $f\left(\pm\frac{2}{\sqrt{3}}\right) = \frac{16}{9} - \frac{32}{3} + 3 = -\frac{53}{9}$: inflections at $\left(\pm\frac{2}{\sqrt{3}}, -\frac{53}{9}\right)$.

d. Critical points in $(-1, 3)$: $x = 0$ ($f = 3$) and $x = 2$ ($f = -13$). The ends: $f(-1) = 1 - 8 + 3 = -4$ and $f(3) = 81 - 72 + 3 = 12$. **Greatest 12, at $x = 3$; least -13, at $x = 2$.** The greatest value is at an end, not at the hilltop $x = 0$.

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MARKING SCHEME A · 8

✓ a. $f'(x) = 4x(x - 2)(x + 2)$.

✓ a. Increasing on $(-2, 0)$, $(2, \infty)$; decreasing on $(-\infty, -2)$, $(0, 2)$.

✓ b. Local maximum 3 at $x = 0$ ($f'' < 0$).

✓ b. Local minimum -13 at $x = \pm 2$ ($f'' > 0$).

✓ c. $f'' = 0$ at $x = \pm\frac{2}{\sqrt{3}}$, with the concavity on each interval.

✓ c. Inflections at $\left(\pm\frac{2}{\sqrt{3}}, -\frac{53}{9}\right)$.

✓ d. $f(-1) = -4$, $f(3) = 12$ compared with 3 and -13.

✓ d. Greatest 12 at $x = 3$, least -13 at $x = 2$.

6 Think

ANSWER HA · 3

No. Let the two sides at right angles to the wall be x m; the side along the wall is $60 - 2x$ m. Area $A = x(60 - 2x) = 60x - 2x^2$, $A' = 60 - 4x = 0$ at $x = 15$, and $A'' = -4 < 0$: a maximum. The best pen is **15 m by 30 m**, the long side along the wall, 450 m^2, against 400 m^2 for the square. The square is best only when the fence goes all the way round; here the wall is a free fourth side.

MARKING SCHEME

✓ The area $A = x(60 - 2x)$ with the wall as one side.

✓ $A' = 0$ at $x = 15$, a maximum.

✓ 15 m by 30 m, 450 m^2, more than the square's 400 m^2: Ankit is wrong.

7 Activity: Slopes with a ruler

(not marked)

WHAT STUDENTS SHOULD FIND

The derivative gives slopes 2, 4 and 6. Careful tangents come within about 10% of them; the error is in placing the ruler, since many lines look as if they touch. That is why the derivative, found from the formula, is used rather than a drawing: it gives the slope exactly.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	4	11	14	4	**33**

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