Unit 17 · Limits and Continuity

HARDER PRACTICE QUESTIONS

Full marks 28

Write your answers on separate paper. Show your working. Angles are in radians unless degrees are given.

1 A join that no constant can mend [4]

A function is given in three pieces:

$f(x)$	when
$x^2 + a$	$x < 1$
b	$x = 1$
$2x + a^2$	$x > 1$

a. Show that no values of a and b make f continuous at $x = 1$. [2]

b. The size of the jump at $x = 1$ depends on a. Which a makes it smallest, and how small is it then? [1]

c. Sampada says: "Take b as the average of the two one-sided limits. Then $f(1)$ sits in the middle and the gap is closed." Is she right? [1]

2 When every value gives 0/0 [5]

$\lim_{x \to 0} \dfrac{a \sin x - \sin 2x}{x^3}$ is a finite number.

a. Explain why "the top must be 0 at $x = 0$" doesn't help to find a. [1]

b. Find a and the value of the limit. [3]

c. What happens to the expression as $x \to 0$ when $a = 3$? [1]

3 Three tugs of war at infinity [6]

Nuwan wrote these three answers. For each, name the form, say why his reason fails, and find the right limit.

i. $\lim_{x \to \infty} \left(\dfrac{x+3}{x-1} \right)^x = 1$, "because the base tends to 1". [2]

ii. $\lim_{x \to \infty} \left(\sqrt{x^2 + 5x} - \sqrt{x^2 + x} \right) = 0$, "because both roots are about x". [2]

iii. $\lim_{x \to \infty} x\left(e^{1/x} - 1 \right) = 0$, "because $e^{1/x} - 1$ tends to 0". [2]

4 Two jumps that cancel [4]

$f(x) = [x] + [-x]$, where $[x]$ is the greatest integer function.

a. Find $f(2.5)$, $f(-0.3)$ and $f(2)$. [1]

b. Find the left-hand and right-hand limits of f at $x = 2$, and name the type of discontinuity there. [2]

c. A function g is equal to f except at the integers, and is continuous for every x. What is g? [1]

5 Which bank? [5]

Sudarshan has Rs 1,00,000 to save for one year. Bank A pays 10% a year, added every quarter. Bank B pays 9.9% a year, added continuously, so that P grows to Pe^r in a year. (The rates are made up for the question.)

a. Find what each bank gives after the year, to the nearest rupee. Which is better? [3]

b. What yearly rate, added continuously, would give exactly what Bank A gives? Use it to check your answer to a. [2]

6 How small is a small angle? [4]

The proof of $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ uses $\cos \theta < \frac{\sin \theta}{\theta} < 1$ for $0 < \theta < \frac{\pi}{2}$.

a. Use it to show that when $\cos \theta \geq 0.99$, $\sin \theta$ is within 1% of θ. Up to what angle, in degrees, does this guarantee it? [2]

b. Yangchen works out $\frac{\sin \theta}{\theta}$ for $\theta = 14°$ on a calculator and gets 0.9901: still within 1%. Does this contradict your answer to a.? Explain what a. says and what it doesn't. [2]

Science Playground, Katunje, Bhaktapur
https://science-playground.com/learn/grade-11-math/limits-and-continuity