Unit 17 · Limits and Continuity

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 28

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 A join that no constant can mend

ANSWER A · 4

Tests: continuity needs LHL = RHL before $f(a)$ matters; the routine "set the pieces equal and solve" can have no solution. The tempting wrong answers are to "find" a from $a^2 - a + 1 = 0$ anyway (a slip in the quadratic formula), or to think a value of $f(1)$ can mend a jump.

a. LHL $= 1 + a$ (first piece), RHL $= 2 + a^2$ (third piece). For continuity they must be equal: $1 + a = 2 + a^2$, that is $a^2 - a + 1 = 0$, whose discriminant is $1 - 4 = -3 < 0$: no real a. So the limit at 1 never exists, and no b can help.

b. RHL − LHL $= (2 + a^2) - (1 + a) = a^2 - a + 1 = \left(a - \frac{1}{2}\right)^2 + \frac{3}{4}$. It is smallest, $\mathbf{\frac{3}{4}}$, at $a = \mathbf{\frac{1}{2}}$, and never 0: the jump can be made smaller, never closed.

c. **No.** $f(1)$ plays no part in the one-sided limits, which are $1 + a$ and $2 + a^2$ whatever b is. With LHL ≠ RHL there is no limit at 1, so f has a jump discontinuity for every b.

MARKING SCHEME

✓ LHL $= 1 + a$ and RHL $= 2 + a^2$.

✓ $a^2 - a + 1 = 0$ has no real root (discriminant -3), so no a, b works.

✓ The jump $\left(a - \frac{1}{2}\right)^2 + \frac{3}{4}$ is smallest, $\frac{3}{4}$, at $a = \frac{1}{2}$.

✓ Sampada is wrong: $f(1)$ doesn't change the one-sided limits.

2 When every value gives 0/0

ANSWER HA · 5

Tests: a parameter fixed by asking for a finite limit, when the easy condition (top 0 where the bottom is 0) holds for every value; the student has to see how fast the top shrinks. The tempting wrong answer is that a can be anything, since every a gives $\frac{0}{0}$.

a. $\sin 0 = 0$, so the top is 0 at $x = 0$ for every a: the condition is true whatever a is, and fixes nothing.

b. $a \sin x - \sin 2x = a \sin x - 2 \sin x \cos x = \sin x(a - 2 \cos x)$, so $\frac{a \sin x - \sin 2x}{x^3} = \frac{\sin x}{x} \cdot \frac{a - 2\cos x}{x^2}$. As $x \to 0$, $\frac{\sin x}{x} \to 1$ and $a - 2\cos x \to a - 2$. If $a \neq 2$, that is a number other than 0 divided by $x^2 \to 0$: no finite limit. So $a = \mathbf{2}$, and then $\frac{\sin x}{x} \cdot \frac{2(1 - \cos x)}{x^2} \to 1 \cdot 2 \cdot \frac{1}{2} = \mathbf{1}$.

c. $\frac{\sin x}{x} \cdot \frac{3 - 2\cos x}{x^2}$: the top tends to 1 and x^2 to 0 through positive values, so the expression **grows without bound** from both sides: no finite limit.

MARKING SCHEME

✓ a. The top is 0 at $x = 0$ for every a.

✓ b. $a \sin x - \sin 2x = \sin x(a - 2 \cos x)$.

✓ b. A finite limit needs $a - 2\cos x \to 0$: $a = 2$.

✓ b. The limit is $2 \cdot \frac{1}{2} = 1$.

✓ c. With $a = 3$, about $\frac{1}{x^2}$: grows without bound.

3 Three tugs of war at infinity

ANSWER

Tests: the three indeterminate forms 1^∞, $\infty - \infty$ and $0 \times \infty$, each needing its own rewriting; the tempting answers are Nuwan's, each treating one side of the tug of war as if the other weren't there.

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i. 1^∞: the base tends to 1 but the power grows without bound. $\frac{x+3}{x-1} = 1 + \frac{4}{x-1}$, so $\left(1+\frac{4}{x-1}\right)^x = \left(1+\frac{4}{x-1}\right)^{x-1}\left(1+\frac{4}{x-1}\right) \to e^4 \cdot 1 = e^4$. A · 6

ii. $\infty - \infty$: "about x" throws away exactly the parts that differ. Multiply and divide by $\sqrt{x^2+5x} + \sqrt{x^2+x}$: $\frac{4x}{\sqrt{x^2+5x}+\sqrt{x^2+x}} = \frac{4}{\sqrt{1+\frac{5}{x}}+\sqrt{1+\frac{1}{x}}} \to \frac{4}{2} = \mathbf{2}$.

iii. $0 \times \infty$: the bracket shrinks but x grows. Put $u = \frac{1}{x}$, so $u \to 0^+$: $x\left(e^{1/x}-1\right) = \frac{e^u - 1}{u} \to \mathbf{1}$.

MARKING SCHEME

- ✓ i. 1^∞: the power grows too.
- ✓ i. e^4.
- ✓ ii. $\infty - \infty$, rewritten with the conjugate.
- ✓ ii. 2.
- ✓ iii. $0 \times \infty$, rewritten with $u = \frac{1}{x}$ as $\frac{e^u - 1}{u}$.
- ✓ iii. 1.

4 Two jumps that cancel

ANSWER HA · 4

Tests: classifying by the one-sided limits rather than by what the parts do. The tempting wrong answer is "a jump, because $[x]$ jumps at 2": here $[x]$ jumps up by 1 and $[-x]$ down by 1, and the jumps cancel.

a. $f(2.5) = 2 + (-3) = -1$; $f(-0.3) = -1 + 0 = -1$; $f(2) = 2 + (-2) = 0$.

b. Just left of 2 ($1 < x < 2$): $[x] = 1$ and $-2 < -x < -1$, so $[-x] = -2$: $f(x) = -1$. Just right of 2 ($2 < x < 3$): $[x] = 2$, $[-x] = -3$: $f(x) = -1$. LHL = RHL = -1, but $f(2) = 0$: **a removable discontinuity**.

c. Away from the integers $f(x) = -1$, so $\mathbf{g(x) = -1}$ for every x.

MARKING SCHEME

- ✓ a. -1, -1 and 0.
- ✓ b. LHL $= -1$ and RHL $= -1$.
- ✓ b. Removable, since $f(2) = 0 \neq -1$.
- ✓ c. $g(x) = -1$ for every x.

5 Which bank?

ANSWER A · 5

Tests: using the limit $\left(1+\frac{r}{n}\right)^n \to e^r$ to compare two ways of adding interest, where the lower rate wins. The tempting wrong answer is "Bank A, since 10% is more than 9.9%".

a. Bank A: $100000\left(1+\frac{0.10}{4}\right)^4 = 100000 \times 1.025^4 \approx$ **Rs 1,10,381**. Bank B: $100000\, e^{0.099} \approx$ **Rs 1,10,407**. **Bank B is better**, by about Rs 25: adding interest continuously gains more than the 0.1% of rate it gives up.

b. $e^r = 1.025^4$, so $r = 4\ln 1.025 \approx 0.09877$: **about 9.877%**. Bank B's 9.9% is more than that, so Bank B is better, as found in a.

MARKING SCHEME

- ✓ a. Bank A: 100000×1.025^4.
- ✓ a. Rs 1,10,381.
- ✓ a. Bank B: Rs 1,10,407, so Bank B is better.
- ✓ b. $e^r = 1.025^4$, $r = 4\ln 1.025$.
- ✓ b. About 9.877%, less than 9.9%: the same verdict.

6 How small is a small angle?

ANSWER

Tests: reading the sandwich inequality as a guarantee (a condition that is enough), not as the exact place where the 1% is reached. The tempting wrong answer is that 8.1° is where the error becomes 1%.

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a. $\theta - \sin\theta = \theta\left(1 - \dfrac{\sin\theta}{\theta}\right) < \theta(1 - \cos\theta) \le 0.01\,\theta$ when $\cos\theta \ge 0.99$. And $\cos\theta \ge 0.99$ for $\theta \le \cos^{-1} 0.99 \approx 0.1415$ radians, about **8.1**°.

b. **No.** a. says that every angle up to 8.1° is within 1%; it says nothing about larger angles, which may be within 1% too. The inequality isn't tight: $\cos\theta$ is well below $\dfrac{\sin\theta}{\theta}$ (at 8°, $1 - \cos\theta \approx 0.0097$ but $1 - \dfrac{\sin\theta}{\theta} \approx 0.0032$, about a third). The true 1% point is near 14°, which is what Yangchen found.

MARKING SCHEME

✓ a. $1 - \dfrac{\sin\theta}{\theta} < 1 - \cos\theta \le 0.01$.

✓ a. $\cos^{-1} 0.99 \approx 0.1415$ rad, about 8.1°.

✓ b. No contradiction: a. is a guarantee up to 8.1°, not the boundary.

✓ b. The inequality isn't tight ($\cos\theta$ is well below $\dfrac{\sin\theta}{\theta}$), so the true 1% point is larger, near 14°.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	15	13	**28**

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