Unit 17 · Limits and Continuity

ANSWER KEY · FOR TEACHERS

Full marks 34

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 3

1. left-hand limit
2. indeterminate form
3. continuous
4. removable
5. jump
6. sandwich theorem

Each carries 0.5 mark.

2 True or false

U · 3

i. ✗ **False**: The limit is about values near a, never at it: $\dfrac{x^2-1}{x-1}$ has no value at 1 but its limit there is 2.

ii. ✓ **True**

iii. ✗ **False**: Only in radians; with θ in degrees the limit is $\dfrac{\pi}{180}$.

iv. ✓ **True**

v. ✗ **False**: $\infty - \infty$ is indeterminate: $\sqrt{x^2+x} - x \to \dfrac{1}{2}$ as $x \to \infty$, while $x^2 - x$ grows without bound.

vi. ✓ **True**

Each carries 0.5 mark.

3 Multiple choice

i. **b.** 0^0 — K · 1

In 0^0 the base shrinks towards 0 and the power towards 0, and the two pull opposite ways, so the answer depends on the functions. $\dfrac{0}{\infty}$ and 0^∞ are both 0, and $\infty \times \infty$ grows without bound.

ii. **b.** $\dfrac{5}{2}$ — U · 1

$\dfrac{\sin 5x}{5x} \cdot \dfrac{2x}{\tan 2x} \cdot \dfrac{5x}{2x} \to 1 \cdot 1 \cdot \dfrac{5}{2} \cdot \dfrac{2}{5}$ turns it over; 1 forgets that the angles differ; 10 multiplies the 5 and the 2.

iii. **c.** $\ln 3$ — U · 1

$\dfrac{a^x - 1}{x} \to \ln a$, so the limit is $\ln 3 \approx 1.0986$. 1 is the limit for e^x only; $\dfrac{1}{\ln 3}$ is $\log_3 e$, the limit of $\dfrac{\log_3(1+x)}{x}$.

iv. **b.** a removable discontinuity — U · 1

For $x \neq 3$, $f(x) = x - 2 \to 1$ from both sides, but $f(3) = 2$: the limit exists and isn't $f(3)$. Redefining $f(3) = 1$ removes it. Both sides head for the same 1, so it isn't a jump.

v. **d.** $e^{3/2}$ — A · 1

$\left(1 + \dfrac{x}{2}\right)^{3/x} = \left[\left(1 + \dfrac{x}{2}\right)^{2/x}\right]^{3/2} \to e^{3/2}$. 1 is the 1^∞ trap; e^6 multiplies the 2 and the 3 instead of dividing; $e^{2/3}$ divides the wrong way.

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking

vi. **c.** $f(x) = x[x]$ HA · 1

$x[x]$: just left of 0, $[x] = -1$, so $f(x) = -x \to 0$; just right of 0, $[x] = 0$, so $f(x) = 0$; and $f(0) = 0$. LHL = RHL = $f(0)$: continuous, though $[x]$ itself jumps there (the jump is multiplied by an x that shrinks to 0). $[x]$ and $\frac{|x|}{x}$ jump at 0 (from -1 to 0, from -1 to 1); $\frac{\sin x}{x} \to 1 \neq f(0)$ is removable.

4 Label the drawing

1. $\sin\theta$ 2. $\tan\theta$ U · 1

Each carries 0.5 mark.

5 Short questions

a. ANSWER A · 5

a. $\frac{0}{0}$. Divide top and bottom by $x - 2$ and use $\frac{x^n - a^n}{x - a} \to na^{n-1}$: $\frac{4 \cdot 2^3}{3 \cdot 2^2} = \frac{32}{12} = \mathbf{\frac{8}{3}}$. (Or factorize: $\frac{(x+2)(x^2+4)}{x^2+2x+4} \to \frac{4 \times 8}{12}$.)

b. $\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$, so $\cos 2x - \cos 4x = 2 \sin 3x \sin x$.

$\frac{2 \sin 3x \sin x}{x^2} = 2 \cdot 3 \cdot \frac{\sin 3x}{3x} \cdot \frac{\sin x}{x} \to 6 \cdot 1 \cdot 1 = \mathbf{6}$.

MARKING SCHEME

✓ a. The $x - 2$ taken out (the standard limit, or factorizing).

✓ a. $\frac{8}{3}$.

✓ b. $\cos 2x - \cos 4x = 2 \sin 3x \sin x$.

✓ b. Written as $6 \cdot \frac{\sin 3x}{3x} \cdot \frac{\sin x}{x}$.

✓ b. 6.

b. ANSWER U · 5

a. $x^2 - 4 = 0$ at $x = 2$ and $x = -2$; f is continuous everywhere else. For $x \neq 2$, $f(x) = \frac{(x+3)(x-2)}{(x+2)(x-2)} = \frac{x+3}{x+2}$. At $x = 2$: the limit is $\frac{5}{4}$, but $f(2)$ is not defined: **removable**. At $x = -2$: the top tends to 1 and the bottom to 0, so $f(x) \to +\infty$ from the right and $-\infty$ from the left: **infinite**.

b. LHL $= g(2) = 2k + 1$; RHL $= 4 - k$. Continuity needs $2k + 1 = 4 - k$: $k = \mathbf{1}$.

MARKING SCHEME

✓ a. Discontinuous at $x = 2$ and $x = -2$ only.

✓ a. $x = 2$: the limit $\frac{5}{4}$ exists, $f(2)$ undefined: removable.

✓ a. $x = -2$: infinite.

✓ b. LHL $= g(2) = 2k + 1$ and RHL $= 4 - k$.

✓ b. $k = 1$.

6 Long question

ANSWER

a. Triangle OAP lies inside sector OAP, which lies inside triangle OAT, so $\frac{1}{2} \cdot 1 \cdot \sin\theta < \frac{1}{2} \cdot 1^2 \cdot \theta < \frac{1}{2} \cdot 1 \cdot \tan\theta$ (half base times height; a sector is $\frac{1}{2}r^2\theta$, θ in radians). Multiply by $\frac{2}{\sin\theta} > 0$: $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$. Turn each over (all positive, so the order reverses): $\cos\theta < \frac{\sin\theta}{\theta} < 1$. As $\theta \to 0^+$, $\cos\theta \to 1$, so $\frac{\sin\theta}{\theta}$ is squeezed to 1 (the sandwich theorem).

b. $\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta}$: the values to the left of 0 are the same as those to the right, so LHL = RHL = 1.

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking

c. $1 - \cos 4x = 2\sin^2 2x$, so $f(x) = \dfrac{2\sin^2 2x}{x \sin x} = 2 \cdot 4 \cdot \left(\dfrac{\sin 2x}{2x}\right)^2 \cdot \dfrac{x}{\sin x} \to 8 \cdot 1 \cdot 1 = 8$. Continuity needs $f(0)$ to equal the limit: $k = \mathbf{8}$. **A · 8**

d. $\dfrac{0}{0}$, and x doesn't tend to 0. Put $y = \pi - x$, so $y \to 0$, and $\sin x = \sin(\pi - y) = \sin y$:

$\lim_{y \to 0} \dfrac{\sin y}{y} = \mathbf{1}$.

MARKING SCHEME

- ✓ a. The three regions in order, with their areas $\frac{1}{2} \sin\theta < \frac{1}{2}\theta < \frac{1}{2} \tan\theta$.
- ✓ a. Divided by $\frac{1}{2} \sin\theta$ and turned over, to $\cos\theta < \dfrac{\sin\theta}{\theta} < 1$.
- ✓ a. $\cos\theta \to 1$, so the sandwich theorem gives 1.
- ✓ b. $\dfrac{\sin\theta}{\theta}$ is unchanged when θ changes sign.
- ✓ c. $1 - \cos 4x = 2\sin^2 2x$, written with $\dfrac{\sin 2x}{2x}$ and $\dfrac{x}{\sin x}$.
- ✓ c. $k = 8$.
- ✓ d. $y = \pi - x$ and $\sin x = \sin y$.
- ✓ d. 1.

7 Think

ANSWER **HA · 3**

For example:

$f(x)$	when
x	$x < 2, x \neq 1$
5	$x = 1$
$x + 1$	$x \geq 2$

$f(0) = 0$. Each piece is a polynomial, so the only places to check are 1 and 2. At $x = 1$: LHL = RHL = 1, but $f(1) = 5$: removable (redefining $f(1) = 1$ mends it). At $x = 2$: LHL = 2, RHL = $f(2) = 3$: a jump of $3 - 2 = 1$. Any function that does the same gets full marks.

MARKING SCHEME

- ✓ A function in pieces with $f(0) = 0$, continuous away from 1 and 2.
- ✓ At 1: the limit exists but isn't $f(1)$ (or $f(1)$ undefined), shown.
- ✓ At 2: RHL − LHL = 1, shown.

8 Activity: Closing in on ln 2

(not marked)

WHAT STUDENTS SHOULD FIND

0.717735, 0.695555, 0.693387 from the right and 0.692907, 0.690750, 0.669670 from the left: falling from above and rising from below to the same number, ln 2 = 0.693147..., as $\dfrac{a^x - 1}{x} \to \ln a$ says. The left- and right-hand values close in on it from opposite sides, so the limit exists.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	4	12	14	4	**34**

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking