Unit 14 · Vectors

HARDER PRACTICE QUESTIONS

Full marks 30

Write your answers on separate paper. Show your working.

1 When coefficients may be compared [4]

Mingma is given $(x+y)\vec{a}+(x-2)\vec{b}=3\vec{a}+\vec{b}$. He writes: "Comparing the coefficients, $x+y=3$ and $x-2=1$, so $x=3$, $y=0$, and these are the only values."

a. When $\vec{a}$ and $\vec{b}$ are non-collinear, is he right? Give the reason that allows his step. [1]

b. Now take $\vec{a}=\vec{i}+2\vec{j}-\vec{k}$ and $\vec{b}=-2\vec{i}-4\vec{j}+2\vec{k}$. Find every pair (x, y) that satisfies the equation, and say which part of Mingma's reasoning breaks. [3]

2 Three vectors of squares [6]

$\vec{p}=(1, 1, 1)$, $\vec{q}=(1, 2, 4)$ and $\vec{r}=(1, 3, 9)$: the numbers $1, t, t^2$ for $t=1, 2, 3$.

a. Show that $\vec{p}, \vec{q}, \vec{r}$ are linearly independent. [1]

b. Express $(1, 4, 16)$ as a linear combination of them. [2]

c. Without solving any equations, explain why, for every vector $(1, s, u)$, the three coefficients add up to 1. [1]

d. Find the condition on s and u for $(1, s, u)$ to be a combination of $\vec{q}$ and $\vec{r}$ alone, and say where the tips of all such vectors lie when they are drawn from O. [2]

3 Where two lines in a triangle cross [5]

In triangle OAB, $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$. P is the point of OA with $OP:PA=2:1$, and Q is the midpoint of AB. The lines OQ and BP cross at R.

a. Write $\overrightarrow{OR}$ in two ways, one from each line. [2]

b. Find the ratios $OR:RQ$ and $BR:RP$. [3]

4 A point in the plane of three points [5]

$\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, and they are the position vectors of the points A, B, C.

a. Show that the point P with position vector $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$ lies in the plane through A, B, C exactly when $\alpha+\beta+\gamma=1$. [3]

b. Find λ so that the point with position vector $2\vec{a}-3\vec{b}+\lambda\vec{c}$ is in that plane. [1]

c. Sujana tests b by asking when $\vec{a}, \vec{b}, \vec{c}$ and $2\vec{a}-3\vec{b}+\lambda\vec{c}$ are coplanar vectors. Why can't that test find λ? [1]

5 Two products from one reaction [5]

In photosynthesis the balance was fixed up to a multiple. Suppose instead that hydrogen and oxygen give a mixture of water and hydrogen peroxide (a made-up case for the problem):

$$x\mathrm{H_2}+y\mathrm{O_2}\to z\mathrm{H_2O}+w\mathrm{H_2O_2}$$

a. Write each substance as its (hydrogen, oxygen) vector and find x and y in terms of z and w. [2]

b. Give two balances, in whole numbers, that are not multiples of each other. [1]

c. Using dependence, explain why photosynthesis had only one balance (up to a multiple) and this reaction has many. [2]

6 Turning one vector into the next [5]

$\vec{a}, \vec{b}, \vec{c}$ are non-coplanar and k is a real number.

a. Find every k for which $\vec{a} + k\vec{b}$, $\vec{b} + k\vec{c}$ and $\vec{c} + k\vec{a}$ are linearly dependent. [3]

b. For that k, give the relation between the three vectors, and describe them as a picture. [2]