Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 When coefficients may be compared

HA · 4

ANSWER

Tests: comparing coefficients is allowed only when the vectors are independent (14.3, Theorem 1); with collinear vectors one equation in place of two. The tempting wrong answer is $x = 3, y = 0$ as the only pair in b.

a. Yes. Non-collinear $\vec{a}, \vec{b}$ make each vector of their plane in exactly one way, so the two coefficients must match: $x = 3, y = 0$, and no other pair.

b. Here $\vec{b} = -2\vec{a}$: the two are collinear. Write everything as a multiple of $\vec{a}$:
left side $= (x + y)\vec{a} - 2(x - 2)\vec{a} = (-x + y + 4)\vec{a}$; right side $= 3\vec{a} - 2\vec{a} = \vec{a}$.
So the equation says only $-x + y + 4 = 1$, that is $\mathbf{y = x - 3}$: infinitely many pairs, such as (3, 0), (0, −3), (5, 2).

His step "compare the coefficients" assumed that a vector is made from $\vec{a}$ and $\vec{b}$ in only one way, which is false for collinear vectors: one vector equation along one line gives one number equation, not two.

MARKING SCHEME

- ✓ a. Right, because non-collinear vectors give each vector in exactly one way (uniqueness).
- ✓ b. $\vec{b} = -2\vec{a}$ noticed (collinear).
- ✓ Both sides written as multiples of $\vec{a}$: $(-x + y + 4)\vec{a} = \vec{a}$.
- ✓ All pairs with $y = x - 3$; comparing coefficients fails because $\vec{a}, \vec{b}$ are collinear.

2 Three vectors of squares

HA · 6

ANSWER

Tests: what independence buys (every vector, in exactly one way, by Theorem 2), and reading the coefficients' equations instead of only solving them. The tempting slips: solving a general system in c, and in d setting up three equations in two unknowns without seeing that c already gives one of them, or describing the tips as "a plane" (two vectors' combinations) when the coefficients must add to 1.

a. $\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} = 1(18 - 12) - 1(9 - 4) + 1(3 - 2) = 6 - 5 + 1 = 2 \neq 0$: **independent**.

b. $x + y + z = 1, x + 2y + 3z = 4, x + 4y + 9z = 16$. Subtracting: $y + 2z = 3$ and $2y + 6z = 12$, so $z = 3$, $y = -3$, $x = 1$. $\mathbf{(1, 4, 16) = \vec{p} - 3\vec{q} + 3\vec{r}}$. Check: $(1 - 3 + 3, 1 - 6 + 9, 1 - 12 + 27)$ ✓.

c. The first component of $x\vec{p} + y\vec{q} + z\vec{r}$ is $x + y + z$, since every first component is 1; it must equal the vector's first component, 1.

d. A combination of $\vec{q}$ and $\vec{r}$ alone has the coefficient of $\vec{p}$ equal to 0, so by c the other two add to 1: $(1, s, u) = (1 - z)\vec{q} + z\vec{r}$. Its components: $s = 2 + z$ and $u = 4 + 5z$, so $\mathbf{u = 5s - 6}$ (for example (1, 4, 14)). By 14.1, $(1 - z)\vec{q} + z\vec{r}$ is the position vector of a point of the line through the tips $Q(1, 2, 4)$ and $R(1, 3, 9)$: **the tips lie on the line** QR, not anywhere in the plane of $\vec{q}$ and $\vec{r}$.

MARKING SCHEME

- ✓ a. The determinant 2, not 0: independent.
- ✓ b. The three equations from the components.
- ✓ $\vec{p} - 3\vec{q} + 3\vec{r}$.
- ✓ c. Every first component is 1, so the coefficients' sum is the first component, 1.
- ✓ d. The coefficients add to 1 (from c): $u = 5s - 6$.
- ✓ The tips on the line through Q and R (14.1).

3 Where two lines in a triangle cross

ANSWER

Tests: a point found from two descriptions, and comparing coefficients because $\vec{a}, \vec{b}$ are non-collinear (two sides of a triangle). The tempting wrong answer is 2 : 1 for both, carried over from the medians.

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a. $\overrightarrow{OP} = \frac{2}{3}\vec{a}$ and $\overrightarrow{OQ} = \frac{1}{2}(\vec{a} + \vec{b})$. **A · 5**

On OQ: $\overrightarrow{OR} = s\overrightarrow{OQ} = \frac{s}{2}\vec{a} + \frac{s}{2}\vec{b}$.

On BP: $\overrightarrow{OR} = \vec{b} + t\left(\frac{2}{3}\vec{a} - \vec{b}\right) = \frac{2t}{3}\vec{a} + (1-t)\vec{b}$.

b. $\vec{a}, \vec{b}$ are non-collinear, so the coefficients agree: $\frac{s}{2} = \frac{2t}{3}$ and $\frac{s}{2} = 1 - t$. So $\frac{2t}{3} = 1 - t$, $t = \frac{3}{5}$, and $s = \frac{4}{5}$.

$OR = \frac{4}{5}OQ$: $OR : RQ = \mathbf{4 : 1}$. $BR = \frac{3}{5}BP$: $BR : RP = \mathbf{3 : 2}$.

(Check: $\overrightarrow{OR} = \frac{2}{5}\vec{a} + \frac{2}{5}\vec{b}$ from both.)

MARKING SCHEME

- ✓ $\overrightarrow{OR} = \frac{s}{2}(\vec{a} + \vec{b})$ on OQ.
- ✓ $\overrightarrow{OR} = \frac{2t}{3}\vec{a} + (1-t)\vec{b}$ on BP.
- ✓ Coefficients compared, with the reason ($\vec{a}, \vec{b}$ non-collinear).
- ✓ $t = \frac{3}{5}, s = \frac{4}{5}$.
- ✓ $OR : RQ = 4 : 1$ and $BR : RP = 3 : 2$.

4 A point in the plane of three points

ANSWER **HA · 5**

Tests: four points are coplanar when the arrows from one of them are (14.2), not when their position vectors are; and comparing coefficients over a basis of three. The tempting wrong method is Sujana's, which tests the position vectors (arrows from O, a point not in the plane) and so can't pick out any λ.

a. $\overrightarrow{AP} = (\alpha - 1)\vec{a} + \beta\vec{b} + \gamma\vec{c}$, $\overrightarrow{AB} = \vec{b} - \vec{a}$, $\overrightarrow{AC} = \vec{c} - \vec{a}$. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are non-collinear (else $\vec{a}, \vec{b}, \vec{c}$ would be dependent), so P is in the plane exactly when $\overrightarrow{AP} = s\overrightarrow{AB} + t\overrightarrow{AC} = -(s+t)\vec{a} + s\vec{b} + t\vec{c}$ for some s, t.
$\vec{a}, \vec{b}, \vec{c}$ are independent, so compare: $s = \beta$, $t = \gamma$, and $\alpha - 1 = -(\beta + \gamma)$. So such s, t exist exactly when $\alpha + \beta + \gamma = 1$. ✓

b. $2 - 3 + \lambda = 1$: $\lambda = \mathbf{2}$.

c. Her four vectors include $\vec{a}, \vec{b}, \vec{c}$, which are non-coplanar, so the four are never coplanar, whatever λ is; and asked as "dependent", four vectors in space always are. Either way the answer doesn't depend on λ. Her vectors are arrows from O, and O is not in the plane of A, B, C (its coefficients add to 0, not 1); the plane of the points is tested by arrows from a point of it.

MARKING SCHEME

- ✓ a. $\overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}$ in terms of $\vec{a}, \vec{b}, \vec{c}$.
- ✓ $\overrightarrow{AP} = s\overrightarrow{AB} + t\overrightarrow{AC}$ as the condition.
- ✓ Coefficients compared (independence) to give $\alpha + \beta + \gamma = 1$, both ways.
- ✓ b. $\lambda = 2$.
- ✓ c. The four are never coplanar ($\vec{a}, \vec{b}, \vec{c}$ aren't) and always dependent, whatever λ: the arrows must start from a point of the plane, not O.

5 Two products from one reaction

ANSWER

Tests: what the uniqueness of the photosynthesis balance rested on (three of the four vectors independent in a space of three numbers), turned round. The tempting wrong answer: "four vectors are dependent, so there is one balance", or a single balance given as the answer.

a. $H_2 = (2, 0)$, $O_2 = (0, 2)$, $H_2O = (2, 1)$, $H_2O_2 = (2, 2)$. Hydrogen: $2x = 2z + 2w$; oxygen: $2y = z + 2w$. So $x = z + w$, $y = \frac{z}{2} + w$.

b. For example $z = 2$, $w = 0$: $2\,H_2 + O_2 \to 2\,H_2O$; and $z = 0$, $w = 1$: $H_2 + O_2 \to H_2O_2$. (Any mix, such as $3\,H_2 + 2\,O_2 \to 2\,H_2O + H_2O_2$, is also a balance.)

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c. Photosynthesis: four vectors of three numbers, and three of them (CO_2, H_2O, O_2) independent, so glucose is their combination in only one way: one free choice, the multiple. Here the vectors have only two numbers, so they lie in a plane, where at most two are independent: H_2 and O_2 already make every vector. Both products are combinations of them, each its own way, so two free choices (z and w) remain, and the balance is not unique. HA · 5

MARKING SCHEME

✓ a. The four vectors.

✓ $x = z + w$ and $y = \frac{z}{2} + w$.

✓ b. Two balances that are not multiples.

✓ c. Photosynthesis: three independent vectors in three numbers leave only the multiple free.

✓ Here: vectors of two numbers, at most two independent, so two free choices.

6 Turning one vector into the next

ANSWER HA · 5

Tests: the determinant test on coefficients over the basis $\vec{a}, \vec{b}, \vec{c}$, and factoring a cubic completely. The tempting wrong answers: "never, as with $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$", or $k = \pm 1$.

a. The coefficients of $\vec{a}, \vec{b}, \vec{c}$ are the rows $(1, k, 0)$, $(0, 1, k)$, $(k, 0, 1)$:

$$\det\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & k \\ k & 0 & 1 \end{pmatrix} = 1(1-0) - k(0-k^2) + 0 = 1 + k^3 = (k+1)(k^2 - k + 1).$$

$k^2 - k + 1$ has discriminant $1 - 4 < 0$, so no real roots. **Only $k = -1$.**

b. At $k = -1$: $(\vec{a} - \vec{b}) + (\vec{b} - \vec{c}) + (\vec{c} - \vec{a}) = \vec{0}$. They are the three sides of triangle ABC (with $\vec{a}, \vec{b}, \vec{c}$ as the corners' position vectors) taken round it: $\overrightarrow{BA} + \overrightarrow{CB} + \overrightarrow{AC} = \vec{0}$, all in the plane of the triangle.

MARKING SCHEME

✓ a. The coefficient determinant set up (valid since $\vec{a}, \vec{b}, \vec{c}$ are independent).

✓ $1 + k^3$.

✓ Factored, $k^2 - k + 1$ has no real root: only $k = -1$.

✓ b. $(\vec{a} - \vec{b}) + (\vec{b} - \vec{c}) + (\vec{c} - \vec{a}) = \vec{0}$.

✓ The sides of a triangle taken round it, coplanar.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	5	25	**30**

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