Unit 14 · Vectors

ANSWER KEY · FOR TEACHERS

Full marks 35

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 3

1. collinear
2. coplanar
3. linear combination
4. dependent
5. independent
6. basis

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: Any two vectors are coplanar; three need not be ($\vec{i}, \vec{j}, \vec{k}$ are not).

iii. ✓ **True**

iv. ✗ **False**: That is true of every set of vectors; dependence needs a combination equal to $\vec{0}$ with some number not zero.

v. ✗ **False**: Any four vectors in space are linearly dependent.

vi. ✓ **True**

Each carries 0.5 mark.

3 Multiple choice

i. **b.** coplanar — K · 1

Three vectors are dependent exactly when one lies in the plane of the other two: coplanar. Collinear vectors are coplanar too, so they are one case of it, not the whole condition.

ii. **b.** -4 — U · 1

The first two components give $\lambda = -2$ ($-6 = -2 \times 3$, $4 = -2 \times (-2)$), so $8 = -2m$ and $m = -4$. 4 forgets the sign of λ, -16 multiplies 8 by λ instead of dividing, and -2 is λ itself.

iii. **d.** $\vec{i} + \vec{j}, \vec{j} + \vec{k}, \vec{i}$ — U · 1

The determinant of (1, 1, 0), (0, 1, 1), (1, 0, 0) is $1(0-0) - 1(0-1) + 0 = 1 \neq 0$. The first two are collinear; $\vec{i} + \vec{j}$ is a combination of $\vec{i}$ and $\vec{j}$; and a set with $\vec{0}$ is always dependent.

iv. **a.** $2\vec{a} + \vec{b}$ — U · 1

$x + 3y = 5$ and $2x - y = 3$ give $x = 2$, $y = 1$. The others make $7\vec{i}$, $-\vec{i} + 5\vec{j}$ and $7\vec{j}$.

v. **d.** 2 — A · 1

$\det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & \lambda \\ 3 & 3 & 1 \end{pmatrix} = 1(1 - 3\lambda) - 2(2 - 3\lambda) - 1(6 - 3) = 3\lambda - 6 = 0$, so $\lambda = 2$. Then the third vector is the sum of the first two.

vi. **a.** $x + y = 3$ — HA · 1

Collinear means $(x-1)\vec{u} + (2-y)\vec{v} = \mu\vec{u} + \mu\vec{v}$ for some μ. $\vec{u}, \vec{v}$ are non-collinear, so the coefficients compare: $x - 1 = \mu = 2 - y$, that is $x + y = 3$. $x = 1, y = 2$ is only the case $\mu = 0$ (the zero vector).

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4 Short answer

MARKING SCHEME U · 2

✓ No: $(0, 4, 2) = 2(0, 2, 1)$, so they are collinear.

✓ Test with one λ for all components, not by dividing.

5 Short questions

a. **ANSWER** A · 5

a. $\overrightarrow{AB} = (2, -1, 3)$, $\overrightarrow{AC} = (6, -3, 9) = 3\overrightarrow{AB}$: collinear, with A in common, so A, B, C lie on one line.

b. $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = (4, -2, 6) = 2\overrightarrow{AB}$, so $AB : BC = \mathbf{1 : 2}$. $|\overrightarrow{AB}| = \sqrt{4 + 1 + 9} = \sqrt{14}$; unit vector $\frac{\mathbf{1}}{\sqrt{\mathbf{14}}}(\mathbf{2}\vec{i} - \vec{j} + \mathbf{3}\vec{k})$.

MARKING SCHEME

✓ $\overrightarrow{AB} = (2, -1, 3)$ and $\overrightarrow{AC} = (6, -3, 9)$.

✓ $\overrightarrow{AC} = 3\overrightarrow{AB}$, with A in common: collinear.

✓ $\overrightarrow{BC} = (4, -2, 6) = 2\overrightarrow{AB}$.

✓ $AB : BC = 1 : 2$.

✓ Unit vector $\frac{1}{\sqrt{14}}(2\vec{i} - \vec{j} + 3\vec{k})$.

b. **ANSWER** A · 5

a. $\det\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = 1(-2 - 1) - 2(4 - 1) - 1(2 + 1) = -3 - 6 - 3 = -12 \neq 0$: **independent**.

b. $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$: $x + 2y + z = 4$, $2x - y + z = -1$, $-x + y + 2z = -1$.

The first minus the second: $-x + 3y = 5$. Twice the second minus the third: $5x - 3y = -1$. Adding: $4x = 4$, so $x = 1$, $y = 2$, and from the first $z = 4 - 1 - 4 = -1$.

$\vec{r} = \vec{a} + \mathbf{2}\vec{b} - \vec{c}$. Check: $(1 + 4 - 1, 2 - 2 - 1, -1 + 2 - 2) = (4, -1, -1)$ ✓.

MARKING SCHEME

✓ The determinant of the components written.

✓ Its value $-12 \neq 0$, so independent.

✓ The three equations from the components.

✓ $x = 1$, $y = 2$, $z = -1$.

✓ $\vec{r} = \vec{a} + 2\vec{b} - \vec{c}$, checked.

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6 Long question

ANSWER A · 8

a. $\vec{a}, \vec{b}, \vec{c}$ are independent, so the determinant test works on the coefficients:

$$\det\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 4 & 3 & \lambda \end{pmatrix} = 1(-\lambda - 9) - 2(2\lambda - 12) - 1(6 + 4) = 5 - 5\lambda.$$

Dependent when $5 - 5\lambda = 0$: $\lambda = \mathbf{1}$.

b. $\vec{w} = x\vec{u} + y\vec{v}$. Compare the coefficients of $\vec{a}$ and $\vec{b}$: $x + 2y = 4$, $2x - y = 3$, so $x = 2$, $y = 1$. Of $\vec{c}$: $-2 + 3 = 1$ ✓. $\vec{w} = \mathbf{2}\vec{u} + \vec{v}$.

c. In triangle ABC, take A as origin, $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{AC} = \vec{c}$. The midpoints of AB and AC are D and E: $\overrightarrow{AD} = \frac{1}{2}\vec{b}$, $\overrightarrow{AE} = \frac{1}{2}\vec{c}$.

$\overrightarrow{DE} = \overrightarrow{AE} - \overrightarrow{AD} = \frac{1}{2}(\vec{c} - \vec{b}) = \frac{1}{2}\overrightarrow{BC}$.

So $\overrightarrow{DE}$ is a positive number times $\overrightarrow{BC}$: they are collinear, so $DE \parallel BC$ (the lines are different, since D is not on BC), and $DE = \mathbf{\frac{1}{2}}BC$. ✓

MARKING SCHEME

- ✓ a. The determinant of the coefficients (with the reason: $\vec{a}, \vec{b}, \vec{c}$ independent).
- ✓ Its value $5 - 5\lambda$.
- ✓ $\lambda = 1$.
- ✓ b. $x + 2y = 4$, $2x - y = 3$.
- ✓ $\vec{w} = 2\vec{u} + \vec{v}$, checked with $\vec{c}$.
- ✓ c. $\overrightarrow{AD} = \frac{1}{2}\vec{b}$ and $\overrightarrow{AE} = \frac{1}{2}\vec{c}$.
- ✓ $\overrightarrow{DE} = \frac{1}{2}(\vec{c} - \vec{b}) = \frac{1}{2}\overrightarrow{BC}$.
- ✓ So parallel (collinear vectors) and half as long.

7 Think

ANSWER HA · 3

No. Dependent means **some** vector of the set is a combination of the others, not every one. Take $\vec{i}, 2\vec{i}, \vec{j}, \vec{k}$: they are dependent, since $2(\vec{i}) - 1(2\vec{i}) + 0\vec{j} + 0\vec{k} = \vec{0}$ with numbers not all zero. But $\vec{k}$ is not a combination of $\vec{i}, 2\vec{i}, \vec{j}$: every such combination has $\vec{k}$-component 0.

MARKING SCHEME

- ✓ No: dependence makes some one of them a combination of the others, not each.
- ✓ An example of four dependent vectors, such as $\vec{i}, 2\vec{i}, \vec{j}, \vec{k}$, with its dependence.
- ✓ A vector of the example that is not a combination of the rest ($\vec{k}$, whose component the others lack).

8 Activity: Balancing a flame

(not marked)

WHAT STUDENTS SHOULD FIND

The vectors are CH_4 (1, 4, 0), O_2 (0, 0, 2), CO_2 (1, 0, 2), H_2O (0, 2, 1): four vectors in space, so a dependence must exist. Carbon $x = z$, hydrogen $4x = 2w$, oxygen $2y = 2z + w$, giving $x = 1$, $y = 2$, $z = 1$, $w = 2$: $CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$. Every other balance is a multiple of it.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	4	8	19	4	**35**

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