Unit 13 · Coordinates in Space

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 30

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 A point from its distances to the axes

ANSWER HA · 6

Tests: the distance from an axis, $\sqrt{y^2+z^2}$ for the X-axis, not $|x|$; and working backwards from distances to a point. The tempting wrong start is to take $|x| = \sqrt{13}$.

a. The foot of the perpendicular from P on the X-axis is $(x, 0, 0)$, so the distance is $\sqrt{y^2+z^2}$; likewise $\sqrt{z^2+x^2}$ and $\sqrt{x^2+y^2}$. Their squares add to $2(x^2+y^2+z^2) = 2\,OP^2$.

b. $2\,OP^2 = 13+10+5 = 28$, so $OP = \sqrt{14}$. Then $x^2 = 14-13 = 1$, $y^2 = 14-10 = 4$, $z^2 = 14-5 = 9$: $P = (\pm 1, \pm 2, \pm 3)$, with any signs: **8 points**, one in each octant.

c. **No.** $2\,OP^2 = 9+16+64 = 89$, so $OP^2 = 44.5$, and then $z^2 = 44.5 - 64 < 0$. In general $z^2 = \frac{1}{2}(d_x{}^2 + d_y{}^2 - d_z{}^2) \geq 0$, so the distance from one axis can never be more than $\sqrt{d_x{}^2 + d_y{}^2}$, the root of the other two squared and added; $8 > \sqrt{9+16} = 5$.

MARKING SCHEME

- ✓ The distance from the X-axis is $\sqrt{y^2+z^2}$ (the foot $(x, 0, 0)$).
- ✓ The three squares add to $2(x^2+y^2+z^2) = 2\,OP^2$.
- ✓ $OP = \sqrt{14}$.
- ✓ $(\pm 1, \pm 2, \pm 3)$: 8 points.
- ✓ $OP^2 = 44.5$ gives $z^2 < 0$.
- ✓ So no such point: one distance can't exceed the root of the sum of the other two squared.

2 Equally far from three points

ANSWER A · 6

Tests: setting up equidistance as equations (squaring turns them linear), and seeing when such a system has no answer. The tempting wrong answer in b is to expect a point every time, or to blame an arithmetic slip.

a. Let $P(x, y, 0)$.

$PA^2 = PB^2$: $(x-4)^2 + (y-1)^2 + 1 = (x-1)^2 + (y-1)^2 + 4$, so $-8x + 17 = -2x + 5$, $x = 2$.

$PA^2 = PC^2$: $(x-4)^2 + (y-1)^2 + 1 = x^2 + (y+3)^2 + 1$, so $-8x - 2y + 17 = 6y + 9$, $8x + 8y = 8$, $y = -1$.

P**(2, −1, 0)**, and $PA^2 = 4+4+1 = 9$, $PB^2 = 1+4+4 = 9$, $PC^2 = 4+4+1 = 9$: the distance is **3**.

b. $PA^2 = PB^2$ gives $x = 2$ as before; $PA^2 = PD^2$: $(x-4)^2 + (y-1)^2 + 1 = (x+2)^2 + (y-1)^2 + 9$ gives $-12x + 4 = 0$, $x = \frac{1}{3}$. The two conditions contradict each other: **no such point**. The reason: AB has direction ratios $-3, 0, 1$ and AD has $-6, 0, 2$, proportional, so A, B, D are collinear. No point can be equally far from three different points on one line (all three would be the same distance from P, so on a ball's surface centered at P, and a straight line meets such a surface in at most two points).

MARKING SCHEME

- ✓ $P(x, y, 0)$ and $PA^2 = PB^2$ gives $x = 2$.
- ✓ $PA^2 = PC^2$ gives $x + y = 1$.
- ✓ $P(2, -1, 0)$.
- ✓ Common distance 3.
- ✓ With D: $x = 2$ and $x = \frac{1}{3}$, a contradiction, so no point.
- ✓ A, B, D are collinear (proportional direction ratios), and no point is equally far from three points of a line.

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3 Where a point lies on a line

ANSWER A · 6

Tests: direction ratios used to move along a line in either direction, the order of points on a line, and the signs of the coordinates that fix an octant. The tempting wrong answer is to put C between A and B, which can't be, since $z = 7$ is outside 3 to -1.

a. Direction ratios of AB: 2, 4, -4. A point of the line is $(1 + 2t, -2 + 4t, 3 - 4t)$, with $t = 0$ at A and $t = 1$ at B; $3 - 4t = 7$ gives $t = -1$: C**(−1, −6, 7)**, on the far side of A from B, **not between** them. Check: $CA = 6$, $AB = 6$, $CB = 12 = CA + AB$, so A is between C and B.

b. YZ-plane ($x = 0$): $t = -\frac{1}{2}$, **(0, −4, 5)**. ZX-plane ($y = 0$): $t = \frac{1}{2}$, **(2, 0, 1)**. XY-plane ($z = 0$): $t = \frac{3}{4}$, $\left(\frac{5}{2}, \mathbf{1}, \mathbf{0}\right)$.

c. In order of t: C (−1), (0, −4, 5) ($-\frac{1}{2}$), A (0), (2, 0, 1) ($\frac{1}{2}$), $\left(\frac{5}{2}, 1, 0\right)$ ($\frac{3}{4}$), B (1). Between the crossings the signs stay fixed: (−, −, +), then (+, −, +), then (+, +, +), then (+, +, −) from the XY-plane on, through B and beyond: the octants **III, IV, I, V** (numbered as in the unit's table).

MARKING SCHEME

- ✓ A general point $(1 + 2t, -2 + 4t, 3 - 4t)$ from the direction ratios.
- ✓ $C(-1, -6, 7)$, not between A and B.
- ✓ (0, −4, 5) and (2, 0, 1).
- ✓ $\left(\frac{5}{2}, 1, 0\right)$.
- ✓ The order C, (0, −4, 5), A, (2, 0, 1), $\left(\frac{5}{2}, 1, 0\right)$, B.
- ✓ Octants III, IV, I, V (or the signs (−, −, +), (+, −, +), (+, +, +), (+, +, −)).

4 Can the angles be like that?

ANSWER HA · 6

Tests: $l^2 + m^2 + n^2 = 1$ as a constraint that the answer must still satisfy. The tempting wrong answer is to stop at $\cos^2\alpha = \frac{3}{5}$, $\alpha \approx 39.2°$, without checking β.

a. $2\cos^2\alpha + \cos^2\beta = 1$, so $\sin^2\beta = 1 - \cos^2\beta = 2\cos^2\alpha$.

b. With $c = \cos^2\alpha$: $2c = 3(1 - c)$, so $c = \frac{3}{5}$: his algebra is right. But then $\cos^2\beta = 1 - 2c = -\frac{1}{5} < 0$: **no real angle** β, so no such line.

c. $2c = k(1 - c)$ gives $c = \frac{k}{k+2}$, and we need $\cos^2\beta = 1 - 2c \geq 0$, so $c \leq \frac{1}{2}$: $2k \leq k + 2$ (with $k + 2 > 0$, since $k \geq 0$: $\sin^2\beta \geq 0$ and $\sin^2\alpha > 0$, as $\alpha = 0°$ would need $2\cos^2\alpha = 2 > 1$), $\mathbf{0} \leq k \leq \mathbf{2}$. (At $k = 2$, $\alpha = 45°$, $\beta = 90°$; at $k = 0$, the Z-axis.) For $k = 1$: $\cos^2\alpha = \frac{1}{3} = \cos^2\beta$, so $l = m = \pm n$: direction ratios 1, 1, 1 (all three angles about 54.7°) or 1, 1, −1 (β about 125.3°, the supplement), a line along one of a cube's long diagonals.

MARKING SCHEME

- ✓ $2\cos^2\alpha + \cos^2\beta = 1$ used to show $\sin^2\beta = 2\cos^2\alpha$.
- ✓ $2c = 3(1 - c)$, $c = \frac{3}{5}$.
- ✓ $\cos^2\beta = -\frac{1}{5} < 0$: no line.
- ✓ $c = \frac{k}{k+2} \leq \frac{1}{2}$.
- ✓ $0 \leq k \leq 2$.
- ✓ $k = 1$: $\cos^2 = \frac{1}{3}$ for all three, direction ratios 1, 1, ± 1, a cube's diagonal.

5 A cable car's straight line

ANSWER

Tests: what a direction cosine means in a real direction (the climb is $90° - \gamma$), moving along a line by its direction ratios, and what stays the same when the climb does. The tempting wrong answer is

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$\gamma \approx 61.9°$, the angle with the vertical.

a. Length $= \sqrt{1200^2 + 900^2 + 800^2} = \sqrt{2{,}890{,}000} =$ **1700 m**. Direction cosines $\frac{12}{17}, \frac{9}{17}, \frac{8}{17}$.

$\gamma = \cos^{-1}\frac{8}{17} \approx 61.9°$ with the vertical, so the climb is $90° - 61.9° \approx$ **28.1**° ($\sin\theta = n = \frac{8}{17}$).

b. A point of the cable is $(1200t, 900t, 800t)$; $1200t = 300$ gives $t = \frac{1}{4}$, and then $y = 225$ ✓, so the cable is over the tower at height $800 \times \frac{1}{4} = 200$ m: **50 m** above the tower's foot.

c. The same climb means the same $n = \frac{8}{17}$, so the same horizontal run for 800 m of height: $\sqrt{1200^2 + 900^2} = 1500$ m ($\tan\theta = \frac{8}{15}$), now all northward: **(0, 1500, 800)**. Its cable is 1700 m too, with direction cosines 0, $\frac{15}{17}, \frac{8}{17}$.

MARKING SCHEME

✓ Length 1700 m.

✓ $n = \frac{8}{17}$, $\gamma \approx 61.9°$.

✓ Climb $90° - \gamma \approx 28.1°$.

✓ Over the tower at $t = \frac{1}{4}$: height 200 m.

✓ 50 m above the tower's foot.

✓ (0, 1500, 800).

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	18	12	**30**

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