Unit 12 · Pair of Straight Lines

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 26

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Passing the test, yet no lines

ANSWER HA · 6

Tests: the condition is necessary, not enough; $h^2 = ab$ means parallel lines, which may be imaginary; the distance between parallel lines from Unit 11. The tempting answer is Diksha's.

a. $a = b = 1$, $h = 1$, $g = f = 1$: $c + 2 - 1 - 1 - c = 0$, whatever c is.

b. The equation is $u^2 + 2u + c = 0$, so $u = -1 \pm \sqrt{1 - c}$: the lines $x + y = -1 + \sqrt{1 - c}$ and $x + y = -1 - \sqrt{1 - c}$, parallel ($h^2 = ab$). Two real lines when $c < 1$; one line, $x + y = -1$, counted twice, when $c = 1$; for $c > 1$ the roots are imaginary and no real point fits: the equation is a pair of imaginary parallel lines, like $x^2 + 1 = 0$.

c. The lines $x + y + 1 \mp \sqrt{1 - c} = 0$ are $\dfrac{2\sqrt{1 - c}}{\sqrt{2}} = \sqrt{2(1 - c)}$ apart. $\sqrt{2(1 - c)} = 2$ gives $1 - c = 2$, $c = -1$: the lines $x + y + 1 = \pm\sqrt{2}$.

MARKING SCHEME

- ✓ a. The values with halves, giving 0 for every c.
- ✓ b. $u = -1 \pm \sqrt{1 - c}$: parallel lines $x + y = -1 \pm \sqrt{1 - c}$.
- ✓ Real and different for $c < 1$.
- ✓ One line at $c = 1$; no real lines for $c > 1$.
- ✓ c. Distance $\sqrt{2(1 - c)}$.
- ✓ $c = -1$.

2 Pairs made to order

ANSWER A · 5

Tests: the bisector and angle formulas read backwards: designing a pair, not recognizing one.

a. With $a = b$, $h(x^2 - y^2) = (a - b)xy = 0$, so the bisectors are $x^2 - y^2 = 0$, $y = \pm x$ ($h \neq 0$, since $h^2 > a^2$). The angle: $\tan\theta = \dfrac{2\sqrt{h^2 - a^2}}{2|a|} = \sqrt{3}$ when $h^2 = 4a^2$, $h = \pm 2a$: $x^2 + 4xy + y^2 = 0$ and $x^2 - 4xy + y^2 = 0$ (or any multiple).

b. The bisectors are $xy = 0$ when $h = 0$: $ax^2 + by^2 = 0$, real only if $ab < 0$. $\tan\theta = \dfrac{2\sqrt{-ab}}{|a + b|} = \sqrt{3}$: with $a = 1$, $4(-b) = 3(1 + b)^2$, $3b^2 + 10b + 3 = 0$, $b = -3$ or $-\dfrac{1}{3}$: $x^2 - 3y^2 = 0$ (lines at $\pm 30^\circ$ to the X-axis) and $3x^2 - y^2 = 0$ (at $\pm 60^\circ$). Two such pairs (up to a multiple): the 60° angle is either the one round the X-axis or the one round the Y-axis.

MARKING SCHEME

- ✓ a. $a = b$ gives $h(x^2 - y^2) = 0$: $y = \pm x$.
- ✓ $h = \pm 2a$.
- ✓ $x^2 + 4xy + y^2 = 0$ and $x^2 - 4xy + y^2 = 0$.
- ✓ b. $h = 0$, $ab < 0$, $\dfrac{2\sqrt{-ab}}{|a + b|} = \sqrt{3}$.
- ✓ $x^2 - 3y^2 = 0$ and $3x^2 - y^2 = 0$: two pairs.

3 A chord seen at a right angle

ANSWER

Tests: making an equation homogeneous (the unit's entrance section, built here step by step); perpendicular when $a + b = 0$; combining with Unit 11's distance. Tempting: forgetting to check that the line meets the circle.

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a. At P and Q both $\frac{x+y}{c}$ and its square are 1, so multiplying $2x+4y$ by one and 4 by the other doesn't change the circle's equation there: P and Q fit the new equation. Every term of it has degree 2, so it is homogeneous: a pair of lines through O (12.2). It holds P and Q, so its lines are OP and OQ.

b. The coefficient of x^2: $1-\frac{2}{c}-\frac{4}{c^2}$; of y^2: $1-\frac{4}{c}-\frac{4}{c^2}$. Perpendicular when they add to 0: $2-\frac{6}{c}-\frac{8}{c^2}=0$, $c^2-3c-4=0$, $c=4$ or $c=-1$.

c. The center is (1, 2), radius $\sqrt{1+4+4}=3$. Distance to $x+y-4=0$: $\frac{|1+2-4|}{\sqrt{2}}\approx 0.71<3$; to $x+y+1=0$: $\frac{|1+2+1|}{\sqrt{2}}=2\sqrt{2}\approx 2.83<3$. Both lines cut the circle, so both values work. (For $c=4$: $P, Q=\left(\frac{3\mp\sqrt{17}}{2}, \frac{5\pm\sqrt{17}}{2}\right)$, and $\overrightarrow{OP}\cdot\overrightarrow{OQ}=0$.)

MARKING SCHEME

- ✓ a. P, Q fit, since $\frac{x+y}{c}=1$ there.
- ✓ Homogeneous, so a pair through O: the lines OP, OQ.
- ✓ b. The x^2 and y^2 coefficients.
- ✓ $c=4$ or $c=-1$.
- ✓ c. Center (1, 2), radius 3.
- ✓ Distances 0.71 and 2.83, both less than 3.

4 Where is the sound?

HA · 5

ANSWER

Tests: a homogeneous pair from a measurement; a sign condition picks half a line; two pairs combined. Tempting: "one pair of microphones is enough".

a. Path difference $34.3\times 0.350\approx 12.0$ cm, so $\cos\varphi=\frac{12}{20}=\frac{3}{5}$: $\frac{x}{\sqrt{x^2+y^2}}=\frac{3}{5}$, squared $25x^2=9x^2+9y^2$, $16x^2-9y^2=0$: $y=\pm\frac{4}{3}x$, with $x>0$ (the right microphone heard first).

b. $34.3\times 0.466\approx 16.0$ cm, $\frac{y}{\sqrt{x^2+y^2}}=\frac{4}{5}$: $25y^2=16x^2+16y^2$, $16x^2-9y^2=0$, the same pair, now with $y>0$.

c. Both conditions: $x>0$ and $y>0$ on $y=\pm\frac{4}{3}x$: the half-line $y=\frac{4}{3}x$, $x>0$, at about 53.1° from the X-axis. One pair gives two mirror-image directions (its bisectors are its own axis and the one across it, since $h=0$); the second pair's sign picks between them.

d. The first pair is $16x^2-9y^2=0$ again; the second gives $\frac{y}{r}=\frac{3}{5}$, $16y^2-9x^2=0$. Lines $y=\pm\frac{4}{3}x$ and $y=\pm\frac{3}{4}x$: no line in common (solving both gives only (0, 0)). No direction in the robot's plane fits both readings: there the angles with the two axes add to 90°, so the squares of the two cosines add to 1, but $\frac{9}{25}+\frac{9}{25}<1$. So the sound came from above or below that plane, or a reading is wrong (an echo).

MARKING SCHEME

- ✓ a. $16x^2-9y^2=0$ with $x>0$.
- ✓ b. The same pair with $y>0$.
- ✓ c. The half-line $y=\frac{4}{3}x$, $x>0$.
- ✓ One pair gives two mirror-image directions; the second sign decides.
- ✓ d. $9x^2-16y^2=0$: no common line, so no direction in the plane fits (sound from above or below, or a wrong reading).

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5 The perpendicular pair's bisectors

ANSWER HA · 4

Tests: using the bisector formula on a pair given in letters; seeing why from the figure.

a. The new pair has $a' = b$, $h' = -h$, $b' = a$: its bisectors are $-h(x^2 - y^2) = (b - a)xy$, which is $h(x^2 - y^2) = (a - b)xy$ multiplied by -1: the same.

b. $6x^2 - 5xy + y^2 = (2x - y)(3x - y)$: $y = 2x$, $y = 3x$. The perpendicular pair: $x^2 + 5xy + 6y^2 = (x + 2y)(x + 3y)$: $y = -\frac{1}{2}x$, $y = -\frac{1}{3}x$. Bisectors of the first: $-\frac{5}{2}(x^2 - y^2) = 5xy$, $x^2 + 2xy - y^2 = 0$; of the second: $\frac{5}{2}(x^2 - y^2) = -5xy$, the same, $x^2 + 2xy - y^2 = 0$.

c. Turning by 90° about O takes each line to the line perpendicular to it through O, so it takes the pair to the perpendicular pair, and the bisectors of the pair to the bisectors of the new pair. But the bisectors are two perpendicular lines, and a quarter turn takes each of them to the other: the pair of bisectors is unchanged.

MARKING SCHEME

- ✓ a. $-h(x^2 - y^2) = (b - a)xy$, the same equation.
- ✓ b. $x^2 + 5xy + 6y^2 = 0$ and the four lines.
- ✓ Both bisector pairs $x^2 + 2xy - y^2 = 0$.
- ✓ c. A quarter turn maps the pair to the perpendicular pair and the (perpendicular) bisectors onto each other.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	5	21	**26**

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