Unit 10 · Trigonometric Equations and General Values

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 31

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 When the rule gives too much

ANSWER HA · 5

Tests: the general-value rule assumes both tangents exist; striking members out of a family; seeing what is left as a simpler family.

a. $3x = n\pi + x$ gives $x = \frac{n\pi}{2}$. For even n, $x = m\pi$: both sides are 0, a solution. For odd n, $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$: neither $\tan x$ nor $\tan 3x$ exists, so these are not solutions. The answer is $x = n\pi$.

(The tempting answer is Ujwal's $\frac{n\pi}{2}$.)

b. $\tan 2x = \frac{1}{\tan x} = \cot x = \tan\left(\frac{\pi}{2} - x\right)$ (with $\tan x \neq 0$), so $2x = n\pi + \frac{\pi}{2} - x$, $x = \frac{n\pi}{3} + \frac{\pi}{6}$. For $n = 1, 4, 7, \ldots$ (n one more than a multiple of 3) this is $\frac{\pi}{2} + m\pi$, where $\tan x$ does not exist: strike them out. What is left, $n = 0, 2, 3, 5, \ldots$, is $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$: $x = n\pi \pm \frac{\pi}{6}$. Check: $\tan\frac{\pi}{3}\tan\frac{\pi}{6} = \sqrt{3}\cdot\frac{1}{\sqrt{3}} = 1$, and $\tan\left(-\frac{\pi}{3}\right)\tan\left(-\frac{\pi}{6}\right) = 1$. ✓

c. The rule comes from $\sin(\theta - \alpha) = 0$ after multiplying by $\cos\theta\cos\alpha$, which assumes both cosines are not 0; so every value where a tangent in the equation has no meaning must be thrown out.

MARKING SCHEME

✓ a. $x = \frac{n\pi}{2}$ from the rule.

✓ The odd multiples of $\frac{\pi}{2}$ rejected (tangents undefined): $x = n\pi$.

✓ b. $x = \frac{n\pi}{3} + \frac{\pi}{6}$.

✓ The members $\frac{\pi}{2} + m\pi$ struck out; the rest is $n\pi \pm \frac{\pi}{6}$.

✓ c. The rule needs both tangents defined; check the values where a cosine is 0.

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2 A constant on the right

ANSWER HA · 5

Tests: a constant on the right is the hidden $k(\sin^2 x + \cos^2 x)$; dividing by $\cos^2 x$ assumes $\cos x \neq 0$.

a. Write $1 = \sin^2 x + \cos^2 x$: $\sin x \cos x = \cos^2 x$, $\cos x\,(\sin x - \cos x) = 0$. $\cos x = 0$: $\frac{\pi}{2}, \frac{3\pi}{2}$. $\tan x = 1$: $\frac{\pi}{4}, \frac{5\pi}{4}$. Four solutions; Sujan kept two and lost $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ (at $\frac{\pi}{2}$: $1 + 0 = 1$ ✓).

b. $2 \sin x \cos x = \cos^2 x$, $\cos x\,(2 \sin x - \cos x) = 0$: $x = (2n+1)\frac{\pi}{2}$ or $\tan x = \frac{1}{2}$, $x = n\pi + \tan^{-1}\frac{1}{2}$.

c. At $x = \frac{\pi}{2}$, $\sin x = 1$, $\cos x = 0$, so the left side is a: a solution exactly when $a = k$ (and then all of $(2n+1)\frac{\pi}{2}$ are). Dividing by $\cos^2 x$ is safe when the coefficient of $\sin^2 x$ is not the constant on the right; when it is, $\cos x$ is a factor and must be taken out.

MARKING SCHEME

✓ a. 1 written as $\sin^2 x + \cos^2 x$, giving $\cos x\,(\sin x - \cos x) = 0$.

✓ $\frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$; two lost.

✓ b. $x = (2n+1)\frac{\pi}{2}$.

✓ $x = n\pi + \tan^{-1}\frac{1}{2}$.

✓ c. The left side is a at $\frac{\pi}{2}$; dividing is safe only when $a \neq k$.

3 Sum and product together

ANSWER HA · 5

Tests: seeing $s = \sin x + \cos x$ and $\sin x \cos x = \frac{s^2 - 1}{2}$ as one unknown; the range of s from $a \cos\theta + b \sin\theta$; a quadratic's least value on an interval (Unit 7). Squaring the equation as it stands is the tempting route, and adds roots.

a. Let $s = \sin x + \cos x$; then $s^2 = 1 + 2 \sin x \cos x$, so $\sin x \cos x = \frac{s^2 - 1}{2}$. The equation is $s = 1 + \frac{s^2 - 1}{2}$, $s^2 - 2s + 1 = 0$, $s = 1$. So $\sqrt{2} \sin\left(x + \frac{\pi}{4}\right) = 1$, $x + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{4}$: $x = 2n\pi$ or $x = 2n\pi + \frac{\pi}{2}$. Check $x = \frac{\pi}{2}$: $1 + 0 = 1 + 0$. ✓

b. $f = s + \frac{s^2 - 1}{2} = \frac{(s+1)^2 - 2}{2}$, with $s = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$ in $[-\sqrt{2}, \sqrt{2}]$. -1 is inside that interval, so the least value is -1, at $s = -1$ (e.g. $x = \pi$: $-1 + 0$). The greatest is at the end farther from -1, $s = \sqrt{2}$: $\frac{(\sqrt{2}+1)^2 - 2}{2} = \frac{1}{2} + \sqrt{2} \approx 1.914$, at $x = \frac{\pi}{4}$. (At $s = -\sqrt{2}$ it is only $\frac{1}{2} - \sqrt{2} \approx -0.914$, not the least.)

MARKING SCHEME

✓ a. $\sin x \cos x = \frac{s^2 - 1}{2}$ and $s = 1$.

✓ $x = 2n\pi$ or $2n\pi + \frac{\pi}{2}$.

✓ b. $f = \frac{(s+1)^2 - 2}{2}$ with $-\sqrt{2} \leq s \leq \sqrt{2}$.

✓ Least -1, at $x = \pi$ (or $\frac{3\pi}{2}$).

✓ Greatest $\frac{1}{2} + \sqrt{2}$, at $x = \frac{\pi}{4}$.

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4 How many solutions?

ANSWER HA · 6

Tests: the range of the equation from a quadratic on $[-1, 1]$; how many x each value of $\sin x$ gives in a turn (two inside $(-1, 1)$, one at ± 1). The tempting answer is "two solutions for every k in the range".

a. $\cos 2x = 1 - 2t^2$: $f(t) = -2t^2 + 2t + 1 = k$. On $[-1, 1]$, f rises to $\frac{3}{2}$ at $t = \frac{1}{2}$ and falls to $f(1) = 1$; $f(-1) = -3$. So solutions exactly when $-3 \le k \le \frac{3}{2}$.

b. $k = \frac{3}{2}$: $t = \frac{1}{2}$ only, $x = \frac{\pi}{6}, \frac{5\pi}{6}$: **2**. $k = 1$: $2t^2 - 2t = 0$, $t = 0$ or 1: $x = 0, \pi$ and $\frac{\pi}{2}$: **3**. $k = 0$: $t = \frac{1 \pm \sqrt{3}}{2}$; $1.366 > 1$ is out, -0.366 gives **2**. $k = -3$: $t = -1$ (and $t = 2$, out): $x = \frac{3\pi}{2}$ only, **1**.

c. Four when both roots t lie strictly inside $(-1, 1)$: both lie in $(0, 1)$ exactly when $1 < k < \frac{3}{2}$ (at $k = 1$ one root reaches $t = 1$). So $1 < k < \frac{3}{2}$.

MARKING SCHEME

- ✓ a. $-2t^2 + 2t + 1 = k$.
- ✓ Range $-3 \le k \le \frac{3}{2}$ (vertex at $t = \frac{1}{2}$, ends $t = \pm 1$).
- ✓ b. $k = \frac{3}{2}$: $\frac{\pi}{6}, \frac{5\pi}{6}$.
- ✓ $k = 1$: $0, \frac{\pi}{2}, \pi$.
- ✓ $k = 0$: 2; $k = -3$: 1.
- ✓ c. $1 < k < \frac{3}{2}$.

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5 Two wires of a three-phase supply

ANSWER HA · 6

Tests: $\sin A = \sin B$ with both A and B containing t, where one parity has no solution; sum to product; reading a fraction of a cycle. Tempting: expecting the wires equal when both are 0, or two families of answers.

a. $100\pi t = n\pi + (-1)^n\left(100\pi t - \frac{2\pi}{3}\right)$. Even n: $0 = n\pi - \frac{2\pi}{3}$, no n works: t cancels and this case gives **nothing**. Odd n: $200\pi t = n\pi + \frac{2\pi}{3}$, $t = \frac{n}{200} + \frac{1}{300}$ with n odd, that is $t = \frac{1}{120} + \frac{m}{100}$. In $[0, 0.02]$: $t = \frac{1}{120}$ s (8.33 ms) and $t = \frac{11}{600}$ s (18.33 ms).

b. At $\frac{1}{120}$: $100\pi t = \frac{5\pi}{6}$, $v_1 = v_2 = 325 \cdot \frac{1}{2} = 162.5$ V. At $\frac{11}{600}$: $100\pi t = \frac{11\pi}{6}$, both -162.5 V. (Not 0: when one wire is at 0 the other is at ∓ 281 V.)

c. $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$ with $A - B = \frac{2\pi}{3}$:

$v_1 - v_2 = 650\sin\frac{\pi}{3}\cos\left(100\pi t - \frac{\pi}{3}\right) = 325\sqrt{3}\cos\left(100\pi t - \frac{\pi}{3}\right)$, a wave of peak $325\sqrt{3} \approx 563$ V, $\sqrt{3}$ times a wire's own peak. It is positive while $-\frac{\pi}{2} < 100\pi t - \frac{\pi}{3} < \frac{\pi}{2}$, half of each turn: $v_1 > v_2$ for **half** of each cycle. This $\sqrt{3}$ is why a 230 V supply has 400 V between two of its phases ($230\sqrt{3} \approx 398$).

Phases 120° apart, line voltage $\sqrt{3}$ times phase voltage (230/400 V): Wikipedia, "Three-phase electric power", read November 2026; 325 V peak, 50 Hz: the unit.

MARKING SCHEME

- ✓ a. Even n gives no solution (explained).
- ✓ Odd n: $t = \frac{1}{120} + \frac{m}{100}$.
- ✓ $\frac{1}{120}$ s and $\frac{11}{600}$ s.
- ✓ b. +162.5 V and −162.5 V.
- ✓ c. $v_1 - v_2 = 325\sqrt{3}\cos\left(100\pi t - \frac{\pi}{3}\right)$, peak about 563 V.
- ✓ Half of each cycle.

6 Make your own

ANSWER HA · 4

Tests: working backwards from solutions to an equation; why $a\cos x + b\sin x = c$ has at most two solutions in a turn.

a. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ are the solutions of $\sin x = \frac{1}{2}$ in $[0, 2\pi]$, and $\frac{\pi}{2}$ of $\sin x = 1$. So $(2\sin x - 1)(\sin x - 1) = 0$, that is $2\sin^2 x - 3\sin x + 1 = 0$. $\sin x = 1$ gives only $\frac{\pi}{2}$ in the interval, $\sin x = \frac{1}{2}$ only $\frac{\pi}{6}$, $\frac{5\pi}{6}$: no others. (Any equation with these solutions and no others is right.)

b. $a\cos x + b\sin x = r\cos(x - \beta)$, so the equation is $\cos(x - \beta) = \frac{c}{r}$, whose solutions are $\beta \pm \alpha$ moved by whole turns: at most two in any turn (three in $[0, 2\pi]$ would need both 0 and 2π, and 0 is not among hers). Only $a = b = 0$ has no r, and then $0 = c$ holds for every x or for none.

MARKING SCHEME

- ✓ a. An equation from the factors $(2\sin x - 1)(\sin x - 1)$.
- ✓ Checked: $\sin x = 1$ once, $\sin x = \frac{1}{2}$ twice in the interval.
- ✓ b. $a\cos x + b\sin x = r\cos(x - \beta)$.
- ✓ So at most two solutions a turn (the endpoints excepted): three impossible.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	0	31	**31**

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