Unit 10 · Trigonometric Equations and General Values

ANSWER KEY · FOR TEACHERS

Full marks 40

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 3

1. identity
2. trigonometric equation
3. general solution
4. principal value
5. periodic
6. homogeneous

Each carries 0.5 mark.

2 True or false

U · 3

i. ✗ **False**: That is the cosine's pattern, and it includes $-\frac{\pi}{6}$, whose sine is $-\frac{1}{2}$. The answer is $\theta = n\pi + (-1)^n \frac{\pi}{6}$.

ii. ✓ **True**

iii. ✗ **False**: It loses the family $x = (2n+1)\frac{\pi}{2}$, where $\cos x = 0$; take $\cos x$ out as a factor instead.

iv. ✗ **False**: Some have none: $\sin\theta = 2$, or $3\sin x + 4\cos x = 6$ (the left side is never more than 5).

v. ✗ **False**: $\theta = 2n\pi$: at π the cosine is -1.

vi. ✓ **True**

Each carries 0.5 mark.

3 Multiple choice

i. **a.** $\theta = n\pi$ — K · 1

The sine is 0 at both ends of the X-axis, 0 and π, and every half turn on. $2n\pi$ misses π; $(2n+1)\frac{\pi}{2}$ are the zeros of the cosine.

ii. **c.** $\theta = n\pi + \alpha$ — K · 1

The tangent repeats every π. $2n\pi \pm \alpha$ is the cosine's rule, $n\pi + (-1)^n\alpha$ the sine's, and $n\pi \pm \alpha$ the rule for $\tan^2\theta = \tan^2\alpha$.

iii. **a.** $\theta = 2n\pi \pm \frac{2\pi}{3}$ — U · 1

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$. $2n\pi \pm \frac{\pi}{3}$ solves $\cos\theta = +\frac{1}{2}$; $n\pi \pm \frac{2\pi}{3}$ and the sine's pattern both include $\frac{\pi}{3}$, whose cosine is $+\frac{1}{2}$.

iv. **b.** $\theta = n\pi \pm \frac{\pi}{3}$ — U · 1

$\tan^2\theta = \tan^2\frac{\pi}{3}$, so $\theta = n\pi \pm \frac{\pi}{3}$: $\tan\theta = \sqrt{3}$ or $-\sqrt{3}$. $n\pi + \frac{\pi}{3}$ forgets $-\sqrt{3}$; $2n\pi \pm \frac{\pi}{3}$ misses $\frac{2\pi}{3}$; $\pm\frac{\pi}{6}$ solves $\tan^2\theta = \frac{1}{3}$.

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v. **b.** $\frac{\pi}{2}$ and $\frac{2\pi}{3}$ **A · 1**

$\cos x(2\cos x + 1) = 0$: $\cos x = 0$ at $\frac{\pi}{2}$, $\cos x = -\frac{1}{2}$ at $\frac{2\pi}{3}$ in $[0, \pi]$. $\frac{4\pi}{3}$ is a solution, but outside the interval; dropping $\cos x = 0$ loses $\frac{\pi}{2}$.

vi. **b.** $\frac{1}{2} \le k \le 1$ **HA · 1**

$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = 1 - \frac{1}{2}\sin^2 2x$. As $\sin^2 2x$ runs over $[0, 1]$, the left side runs over $\left[\frac{1}{2}, 1\right]$: 1 at $x = 0$, $\frac{1}{2}$ at $x = \frac{\pi}{4}$. A sum of two fourth powers is never negative, and here never below $\frac{1}{2}$, so $k = \frac{1}{4}$ has no solution.

4 Short answers

a. **MARKING SCHEME** **K · 3**

- ✓ $\sin x = 1$: $x = (4n + 1)\frac{\pi}{2}$ (or $2n\pi + \frac{\pi}{2}$).
- ✓ $\cos x = 0$: $x = (2n + 1)\frac{\pi}{2}$.
- ✓ $\tan x = 0$: $x = n\pi$.

b. **MARKING SCHEME** **U · 2**

- ✓ $\cos\theta = -\frac{\sqrt{3}}{2} = \cos\frac{5\pi}{6}$.
- ✓ $\theta = 2n\pi \pm \frac{5\pi}{6}$.

c. **MARKING SCHEME** **U · 2**

- ✓ She missed $\cos x = -\frac{1}{2}$.
- ✓ $x = n\pi \pm \frac{\pi}{3}$.

d. **MARKING SCHEME** **A · 2**

- ✓ A member that fails, e.g. $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$, with cosine $-\frac{1}{2}$; an equation it solves, such as $\tan^2 x = 3$.
- ✓ $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

e. **MARKING SCHEME** **A · 3**

- ✓ $\cos x\,(2\sin x - 1) = 0$ (the factor taken out, not divided).
- ✓ $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.
- ✓ $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

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5 Structured question

ANSWER A · 5

a. $r = \sqrt{1+3} = 2$, and $\sin x \cdot \frac{1}{2} + \cos x \cdot \frac{\sqrt{3}}{2} = \sin\left(x + \frac{\pi}{3}\right)$, so the left side is $2\sin\left(x + \frac{\pi}{3}\right)$: solutions exactly when $-2 \le k \le 2$.

b. $\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$, so $x + \frac{\pi}{3} = n\pi + (-1)^n\frac{\pi}{4}$, $x = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{3}$. $n = 1$: $\pi - \frac{\pi}{4} - \frac{\pi}{3} = \frac{5\pi}{12}$; $n = 2$: $2\pi + \frac{\pi}{4} - \frac{\pi}{3} = \frac{23\pi}{12}$ ($n = 0$ gives $-\frac{\pi}{12}$, outside). So $x = \frac{5\pi}{12}, \frac{23\pi}{12}$.

MARKING SCHEME

- ✓ $2\sin\left(x + \frac{\pi}{3}\right)$ (or $2\cos\left(x - \frac{\pi}{6}\right)$).
- ✓ $-2 \le k \le 2$.
- ✓ $\sin\left(x + \frac{\pi}{3}\right) = \sin\frac{\pi}{4}$.
- ✓ $x = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{3}$.
- ✓ $\frac{5\pi}{12}$ and $\frac{23\pi}{12}$.

6 Long question

ANSWER A · 8

a. $\cos\theta - \cos\alpha = -2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$. So $\frac{\theta-\alpha}{2} = n\pi$, $\theta = 2n\pi + \alpha$, or $\frac{\theta+\alpha}{2} = n\pi$, $\theta = 2n\pi - \alpha$: together $\theta = 2n\pi \pm \alpha$. ✓

b. $\cos 7\theta + \cos\theta = 2\cos 4\theta\cos 3\theta$, so $2\cos 4\theta\cos 3\theta = \cos 4\theta$, $\cos 4\theta(2\cos 3\theta - 1) = 0$. $\cos 4\theta = 0$: $4\theta = (2n+1)\frac{\pi}{2}$, $\theta = (2n+1)\frac{\pi}{8}$. $\cos 3\theta = \frac{1}{2}$: $3\theta = 2n\pi \pm \frac{\pi}{3}$, $\theta = \frac{2n\pi}{3} \pm \frac{\pi}{9}$.

c. From the first family $\frac{\pi}{8}$ and $\frac{3\pi}{8}$ ($\frac{5\pi}{8} > \frac{\pi}{2}$); from the second $\frac{\pi}{9}$ ($\frac{2\pi}{3} - \frac{\pi}{9} = \frac{5\pi}{9} > \frac{\pi}{2}$). So $\theta = \frac{\pi}{9}, \frac{\pi}{8}, \frac{3\pi}{8}$.

MARKING SCHEME

- ✓ a. $\cos\theta - \cos\alpha$ as $-2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}$.
- ✓ Each factor 0: $\theta = 2n\pi + \alpha$ and $\theta = 2n\pi - \alpha$.
- ✓ Together $2n\pi \pm \alpha$.
- ✓ b. $\cos 4\theta(2\cos 3\theta - 1) = 0$ (the factor taken out, not divided).
- ✓ $\theta = (2n+1)\frac{\pi}{8}$.
- ✓ $\theta = \frac{2n\pi}{3} \pm \frac{\pi}{9}$.
- ✓ c. $\frac{\pi}{8}$ and $\frac{3\pi}{8}$.
- ✓ $\frac{\pi}{9}$, and no other.

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7 Think

ANSWER HA · 3

Let $t = \sin x$; t can be any number in $[-1, 1]$ and no other. $t^2 - t = \left(t - \frac{1}{2}\right)^2 - \frac{1}{4}$: least $-\frac{1}{4}$ at $t = \frac{1}{2}$, and on $[-1, 1]$ greatest 2 at $t = -1$. So there are solutions exactly when $-\frac{1}{4} \le k \le 2$. She is wrong: $k = -\frac{1}{2}$ has none, and $k = 2$ has $\sin x = -1$, $x = (4n - 1)\frac{\pi}{2}$.

MARKING SCHEME

✓ $t = \sin x$ runs over $[-1, 1]$ only.

✓ $t^2 - t$ has least value $-\frac{1}{4}$ (at $t = \frac{1}{2}$) and greatest 2 (at $t = -1$).

✓ $-\frac{1}{4} \le k \le 2$; Rabina is wrong (e.g. $k = 2$ works, $k = -\frac{1}{2}$ doesn't).

8 Activity: Two patterns side by side

(not marked)

WHAT STUDENTS SHOULD FIND

The column $n\pi + (-1)^n \frac{\pi}{6}$ gives $-\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, every sine $\frac{1}{2}$. The column $2n\pi + \frac{\pi}{6}$ gives $-\frac{11\pi}{6}, \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}$, sines $\frac{1}{2}$, and $2n\pi - \frac{\pi}{6}$ gives $-\frac{13\pi}{6}, -\frac{\pi}{6}, \frac{11\pi}{6}, \frac{23\pi}{6}$, sines $-\frac{1}{2}$. So $n\pi + (-1)^n \frac{\pi}{6}$ solves $\sin\theta = \frac{1}{2}$; $2n\pi \pm \frac{\pi}{6}$ solves $\cos\theta = \frac{\sqrt{3}}{2}$, and half its members have the wrong sine.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	8	9	19	4	**40**

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